An Introduction to Thermal Field Theory

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Abstract

This thesis aims to give an introductory review of thermal field theories. We review the imaginary time formalism to generalize field theories at finite temperature. We study the scalar ϕ^4 theory and gauge theories in a thermal background. We study the propagators and self energies at one-loop approximation and understand how particles acquire thermal masses and the consequences. We show in a hot plasma, there are collective excitation modes absent in the zero temperature case, and a point charge is screened by the thermal effects. However, a naive perturbation theory would break down for soft external momenta owing to the so-called hard thermal loop corrections. Diagrams of higher orders could contribute at the leading order, and hence it is necessary to develop an effective theory. We introduce the basic ideas of the resummation scheme and make several remarks on its applications.

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Contents

1 Introduction: Why Thermal Field Theory?

The conventional quantum field theory is formalized at zero temperature. The theoretical predictions under this framework, for example the cross sections of particle collisions in an accelerator, are extremely good to match experimental data. However, our real world is certainly of non zero temperature, it is natural to ask to what extent the contributions from the temperature start to be relevant? What new phenomena absent at zero temperature could arise in a thermal background?

It may occur to us the heavy-ion collision experiments at LHC and at RHIC. A new state of matter, which is called the quark gluon plasma [1], is predicted to be created in the collisions. The consistence between experimental data and theoretical predictions of the Standard Model has to be verified. A useful theoretical framework to study this hot quark gluon plasma would be the thermal field theory.

We may also think of the early stages of the universe, or astrophysical objects such as white dwarfs and neutron stars, where the temperature is sufficiently high. Thermal field theory would be responsible for our understanding of the phenomena as phase transitions and cosmological inflation in the early universe, the evolution of a neutron star. On the other hand, cosmology and astrophysics are good test fields for theoretical studies to verify practical calculations.

These things that we can think of necessitate formal studies of field theories at finite temperature. This dissertation is devoted to giving an introductory review of thermal field theories. Most of the topics covered in this article are covered in some excellent textbooks [2] [3] [4].

The organization of this article is as follows. We give a brief review of quantum statistical mechanics in Section 2. We then develop the imaginary time formalism to study scalar field and gauge theories at finite temperature in Section 3 and 4, respectively. We will see in Section 5 the breakdown of a naive perturbatiion expansion and the necessity to formulate an effective theory to resum the contributions from the so-called hard thermal loops at all orders.

2 Review of Quantum Statistical Mechanics

In statistical mechanics the concept of *thermal ensembles* is of great importance. The *canonical ensemble* will be particularly useful to describe a system in equilibrium of our interest in this introductory review. The canonical ensemble describes a system in contact with a heat reservoir at a fixed temperature T. Energy can be exchanged between the system and the reservoir, but the particle number N and volume V are fixed. We may also use the *grand canonical ensemble*, where the system exchange both energy and particles with the reservoir at temperature T , while the chemical potential μ of particles and volume V are fixed. We can think of the canonical ensemble as a special case of the grand canonical ensemble in which the particles have vanishing chemical potentials.

In order to formalize quantum field theory at non-zero temperature, for simplicity, we use the canonical ensemble by assuming that the chemical potentials are zero. We will define and calculate quantities such as the density operator, the partition function in terms of the canonical ensemble.

Consider now a dynamical system characterized by a Hamiltonian H. The equilibrium state of the system of volume V is described by the canonical density operator

$$\rho = \exp(-\beta H) \tag{2.1}$$

The *partition function*, a very important quantity playing the central role in our studies of finite temperature field theory, is defined as

$$Z = \mathrm{Tr}\rho \tag{2.2}$$

All thermodynamical quantities can be generated from the partition function. For example,

$$\text{Pressure} \qquad P = T\frac{\partial \ln Z}{\partial V} \tag{2.3}$$

$$\text{Particle number} \qquad N = T\frac{\partial \ln Z}{\partial \mu} \tag{2.4}$$

$$\text{Entropy} \qquad S = T\frac{\partial \ln Z}{\partial T} \tag{2.5}$$

Recall in the zero temperature quantum field theory, the expectation value of a given operator A is

$$\langle A \rangle_0 = \sum_n \langle n|A|n\rangle \tag{2.6}$$

where $|n\rangle$ are a complete set of orthonormal states. However, in a heat bath, the operator expectation should be calculated as the ensemble average with a Boltzmann weight factor

$$\langle A\rangle_\beta = \frac{1}{Z}\sum_n \langle n|A|n\rangle \mathrm{e}^{-\beta H} = \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}A\right) \tag{2.7}$$

or with the use of the density matrix, we write

$$\langle A\rangle_\beta = \frac{\mathrm{Tr}A\rho}{\mathrm{Tr}\rho} \tag{2.8}$$

We can also think of a system characterized by a Hamiltonian H and a set of conserved charges Q with particles of non-zero chemical potential. In this case we shall switch to use the grand canonical ensemble and redefine the density operator

$$\rho = \exp\left[-\beta(H-\mu N)\right] \tag{2.9}$$

The definitions of the other quantities follow similarly as the zero chemical potential case, where we use the canonical ensemble. The use of grand canonical ensemble enables us to extend our studies on cases with non-trivial chemical potentials. The canonical ensemble can be thought of as a special case of the canonical ensemble with vanishing chemical potentials, but there are subtitles to care about.

We are now ready to derive a fundamental relation in finite temperature theory. Consider the two-point correlation function

$$\begin{aligned}
\langle\phi(\mathbf{x},t)\phi(\mathbf{y},0)\rangle_\beta &= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\phi(\mathbf{x},t)\phi(\mathbf{y},0)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\phi(\mathbf{x},t)\mathrm{e}^{-\beta H}\mathrm{e}^{\beta H}\phi(\mathbf{y},0)\mathrm{e}^{-\beta H}\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\phi(\mathbf{x},t)\mathrm{e}^{-\beta H}\mathrm{e}^{i(-i\beta H)}\phi(\mathbf{y},0)\mathrm{e}^{-i(-i\beta H)}\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\phi(\mathbf{x},t)\mathrm{e}^{-\beta H}\phi(\mathbf{y},-i\beta)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\phi(\mathbf{y},-i\beta)\phi(\mathbf{x},t)\right] \\
&= \langle\phi(\mathbf{y},-i\beta)\phi(\mathbf{x},t)\rangle_\beta
\end{aligned} \tag{2.10}$$

where we used the cyclic permutation property of a trace of operator products. We surprisingly see that imaginary temperature plays the role as a time variable. If we define the imaginary time variable

$$\tau = it \qquad t = -i\tau \tag{2.11}$$

then the relation above can be rewritten as

$$\langle\phi(\mathbf{x},\tau)\phi(\mathbf{y},0)\rangle_\beta = \langle\phi(\mathbf{y},\beta)\phi(\mathbf{x},\tau)\rangle_\beta \tag{2.12}$$

This is called *the Kubo-Martin-Schwinger relation*, or the *KMS relation* in short. It follows immediately from this relation that

$$\phi(\mathbf{x}, 0) = \pm\phi(\mathbf{x}, \beta) \tag{2.13}$$

where $\pm$ sign corresponds to whether the fields commute or anti-commute with each other, or in other words, whether the fields are bosonic or fermionic. The KMS relation shows that the fields are periodic or anti-periodic in imaginary time with β.

It is convenient to cope with the fields in the frequency-momentum space. Owing to the periodicity constraint on the fields, the Fourier expansion

$$\phi(\mathbf{x}, \tau) = \sum_n \phi(\mathbf{x}, \omega_n) \mathrm{e}^{i\omega_n \tau} \tag{2.14}$$

is no longer a continuous Fourier integral but a Fourier series instead. In order to satisfy the KMS relation (2.13), we can only take the discrete frequencies

$$\omega_n = \frac{2\pi n}{\beta} \qquad \text{for bosonic fields} \tag{2.15}$$

$$\omega_n = \frac{2\pi(n+1)}{\beta} \qquad \text{for fermionic fields} \tag{2.16}$$

n are integers $-\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty$. These frequencies are called *the Matsubara frequencies*, named after the Matsubara who first formally constructed a thermal field theory in the imaginary time formalism [5].

We also develop a path integral form for the partition function. The advantage of a path integral representation lies in the convenience within this framework to deal with gauge theories than using operator formalism, especially for non-Abelian gauge theories such as QCD. By noting that

$$\mathrm{e}^{-\beta H} = \mathrm{e}^{-i\int_0^{-i\beta} H dt} = \mathrm{e}^{-\int_0^{\beta} H d\tau} \tag{2.17}$$

We may think of $\exp(-\beta H)$ as an evolution operator in imaginary time with $\tau = it$. Recall the standard formalism of path integrals which can be found in many quantum field theory textbooks [6], we have

$$U(q', t'; q, t) = \langle q' | \mathrm{e}^{-iH(t'-t)} | q \rangle = \int \mathcal{D}q(t'') \exp\left[i \int_t^{t'} dt'' \mathcal{L}(t) \right] \tag{2.18}$$

We write down the expression for partition function

$$Z = \int \mathcal{D}\phi \langle \phi | \mathrm{e}^{-\beta H} | \phi \rangle = \int \mathcal{D}\phi \exp\left[-\int_0^{\beta} d\tau \mathcal{L}(\tau) \right] \tag{2.19}$$

All paths $\phi(\mathbf{x}, \tau)$ satisfying the boundary condition (2.13) shall be evaluated in the path integral.

The subtleties of dealing with fermions by introducing anti-commuting Grassmann variables will be discussed more carefully in Section 3.1.

As a brief ending remark of this section, we see that it is very useful to think of the temperature as the imaginary time, but the origin of the correspondence between these two very distinct arguments is an interesting question. This might merely be a coincidence that the evolution operator e^{-iHt} in quantum mechanics is related to the Boltzmann distribution factor $\mathrm{e}^{\beta H}$ in statistical physics by an analytical continuation, but there might be some deeper reasons that we do not well understand.

3 Scalar Field Theory at Finite Temperature

3.1 Free scalar fields

3.1.1 Partition function

We begin with the simplest possible model of a free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{3.1}$$

Our choice of convention for the metric is $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. The conjugate momentum to the field operator is

$$\pi(\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(\mathbf{x}) \tag{3.2}$$

As in the zero temperature case, the field operator and its conjugate momentum can be Fourier expanded in terms of a set of creation and annihilation operators

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left(a_\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + a_\mathbf{k}^\dagger\mathrm{e}^{-i\mathbf{k}\mathbf{x}}\right) \tag{3.3}$$

$$\pi(\mathbf{x}) = -i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{E_\mathbf{k}}{2}}\left(a_\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} - a_\mathbf{k}^\dagger\mathrm{e}^{-i\mathbf{k}\mathbf{x}}\right) \tag{3.4}$$

The equal time commutation relation is imposed as

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab} \tag{3.5}$$

or equivalently

$$[a_\mathbf{p}, a_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{rs} \tag{3.6}$$

We want to compute the partition function to obtain thermodynamicals for free scalar fields. Plug the Lagrangian (3.1) into (2.19), we have

$$Z = \int\mathcal{D}\phi\exp\left\{-i\int_0^{-i\beta}dt\int d^3\mathbf{x}\left[\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right]\right\} \tag{3.7}$$

$$= \int\mathcal{D}\phi\exp\left\{-\int_0^{\beta}d\tau\int d^3\mathbf{x}\left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]\right\} \tag{3.8}$$

To work under the frequency-momentum space, we Fourier expand the fields. Recall that the KMS relation has imposed a periodicity constraint, the Fourier integral we had in the conventional field theories should be replaced by a Fourier series

$$\phi(x) = \sqrt{\frac{\beta}{V}}\sum_n\sum_\mathbf{p}\mathrm{e}^{i(\omega_n\tau+\mathbf{p}\cdot\mathbf{x})}\phi(\omega_n,\mathbf{p}) \tag{3.9}$$

where the allowed Matsubara frequencies are discrete

$$\omega_n = \frac{2\pi n}{\beta} \tag{3.10}$$

Substitute the Fourier expansion (3.9) into the partition function, we obtain

$$\int \mathcal{D}\phi \exp\left\{\frac{\beta}{V}\int_0^\beta d\tau \int d^3\mathbf{x} \sum_{m,\mathbf{k}} \phi_{m,\mathbf{k}} e^{i(\omega_m \tau + \mathbf{k}\cdot\mathbf{x})} \sum_{n,\mathbf{p}} \frac{-\omega_n^2 - \mathbf{p}^2 - m^2}{2} \phi_{n,\mathbf{p}} e^{i(\omega_n \tau + \mathbf{p}\cdot\mathbf{x})}\right\} \tag{3.11}$$

Using the representations for the δ-functions

$$\int_0^\beta d\tau e^{i(\omega_m+\omega_n)\tau} = \beta\delta(m+n) \tag{3.12}$$

$$\int d^3\mathbf{x} e^{i(\mathbf{k}+\mathbf{p})\mathbf{x}} = V\delta^{(3)}(\mathbf{p}+\mathbf{k}) \tag{3.13}$$

We carry out the integration over $d\tau$, $d^3\mathbf{x}$ and then the summation over m, $\mathbf{k}$ with the Kronecker δ-functions, we get

$$\begin{aligned} Z &= \int \mathcal{D}\phi \exp\left\{-\frac{\beta^2}{2}\sum_{n,\mathbf{p}}(\omega_n^2 + \mathbf{p}^2 + m^2)\phi_{-n,-\mathbf{p}}\phi_{n,\mathbf{p}}\right\} \\ &= \int \mathcal{D}\phi \exp\left\{-\frac{\beta^2}{2}\sum_{n,\mathbf{p}}(\omega_n^2 + \mathbf{p}^2 + m^2)\phi^*_{n,\mathbf{p}}\phi_{n,\mathbf{p}}\right\} \\ &= \prod_{n,\mathbf{p}} \int d\phi_n \exp\left\{-\frac{\beta^2}{2}(\omega_n^2 + \mathbf{p}^2 + m^2)\phi^*_{n,\mathbf{p}}\phi_{n,\mathbf{p}}\right\} \\ &= N \cdot \prod_{n,\mathbf{p}} \left(\beta^2(\omega_n^2 + \mathbf{p}^2 + m^2)\right)^{-1/2} \end{aligned} \tag{3.14}$$

Some unimportant integration constant N is independent of temperature and therefore can be dropped.

We are interested in the logarithm of the partition function from which we calculate all the physical measurables. We have

$$\begin{aligned} \ln Z &= -\frac{1}{2}\sum_{n,\mathbf{p}} \ln\left(\beta^2(\omega_n^2 + \omega_\mathbf{p}^2)\right) \\ &= -\frac{1}{2}\sum_{n,\mathbf{p}} \ln\left((2\pi n)^2 + \beta^2\omega_\mathbf{p}^2\right) \\ &= -\frac{1}{2}\sum_{n,\mathbf{p}} \left\{\int_1^{\beta^2\omega_\mathbf{p}^2} \frac{dx^2}{(2\pi n)^2 + x^2} + \ln\left((2\pi n)^2 + 1\right)\right\} \end{aligned} \tag{3.15}$$

The last step can be checked by completing the integral. The reason for doing this is that we can rewrite the frequency sum as a contour integral. The discrete Matsubara frequencies correspond to a collection of poles produced by a well chosen hyperbolic cotangent function on the imaginary axis on the complex plane. The details of this technique is given in the appendix. Using the result (A.7) and dropping terms having no temperature dependence, we obtain

$$
\begin{aligned}
\ln Z &= -\frac{1}{2}\sum_{\mathbf{p}}\int_{1}^{\beta^2\omega_{\mathbf{p}}^2} dx^2 \frac{1}{2x}\left(1+\frac{2}{\mathrm{e}^x-1}\right) \\
&= \sum_{\mathbf{p}}\int_{1}^{\beta\omega_{\mathbf{p}}} dx\left(-\frac{1}{2}-\frac{1}{\mathrm{e}^x-1}\right) \\
&= \sum_{\mathbf{p}}\left\{-\frac{1}{2}\beta\omega_{\mathbf{p}}-\ln\left(1-\mathrm{e}^{-\beta\omega_{\mathbf{p}}}\right)\right\}
\end{aligned}
\tag{3.16}
$$

In the continuum limit, we have $\sum_{\mathbf{p}} \sim V\int d^3\mathbf{p}/(2\pi)^3$, so this can be rewritten as

$$
\ln Z = V\int \frac{d^3\mathbf{p}}{(2\pi)^3}\left\{-\frac{1}{2}\beta\omega_{\mathbf{p}}-\ln\left(1-\mathrm{e}^{-\beta\omega_{\mathbf{p}}}\right)\right\} \tag{3.17}
$$

The first term is nothing but the familiar zero-point energy, which is divergent since it sums over an infinity number of zero-point modes. But we can simply neglect it because the effects of its contribution cannot be measured experimentally.

We have derived the explicit expression for the partition function, now we can easily calculate thermodynamicals from it, for instance, the pressure of the scalar particles.

$$
P = \frac{T}{V}\ln Z = -\frac{1}{\beta}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\ln(1-\mathrm{e}^{-\beta\omega_{\mathbf{p}}}) \tag{3.18}
$$

Take the high energy limit, where $|\mathbf{p}| \gg m$, $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2} \approx |\mathbf{p}|$, we find

$$
\begin{aligned}
P &= \frac{1}{\beta}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(\mathrm{e}^{-\beta\omega_{\mathbf{p}}}+\mathrm{e}^{-2\beta\omega_{\mathbf{p}}}+\mathrm{e}^{-3\beta\omega_{\mathbf{p}}}+\cdots\right) \\
&= \frac{1}{\beta}\int \frac{dp}{2\pi^2}p^2\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{-n\beta\omega_{\mathbf{p}}}}{n} \\
&\approx \frac{1}{\beta}\int \frac{dp}{2\pi^2}p^2\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{-n\beta p}}{n} \\
&= \frac{T^4}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^4} \\
&= \frac{\pi^2}{90}T^4
\end{aligned}
\tag{3.19}
$$

The T^4 behavior can be expected from a dimensional analysis by noting that T is the only characteristic parameter in the free scalar field theory. The pressure is half of the familiar result of black body radiation. This could be understood from our knowledge of statistical mechanics. The mean energy of a system is proportional to the independent degrees of freedom. The real scalar particles are spin zero particles and thus have only one degree of freedom, while the photons have two transverse propagating modes.

3.1.2 Scalar propagators

Let us study the propagators carefully. We define the two-point correlators

$$D^+(x,y) = \langle \phi(x)\phi(y)\rangle_\beta \tag{3.20}$$

$$D^-(x,y) = \langle \phi(y)\phi(x)\rangle_\beta = D^+(y,x) \tag{3.21}$$

We insert complete sets of eigenstates of the Hamiltonian $|n\rangle$'s and express

$$\begin{aligned} D^+(x,y) &= \frac{1}{Z}\sum_{n,m}\langle \mathrm{e}^{-\beta H} n|\phi(x)|m\rangle\langle m|\phi(y)|n\rangle \\ &= \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\langle n|\mathrm{e}^{ipx}\phi(0)\mathrm{e}^{-ipx}|m\rangle\langle m|\mathrm{e}^{ipy}\phi(0)\mathrm{e}^{-ipy}|n\rangle \\ &= \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\mathrm{e}^{i(p_n-p_m)(x-y)}\big|\langle n|\phi(0)|m\rangle\big|^2 \end{aligned} \tag{3.22}$$

which shows that the correlator is a function of $(x-y)$. Its Fourier transform is

$$D^+(k) = \int d^4x \mathrm{e}^{ikx}D^+(x) = \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\big|\langle n|\phi(0)|m\rangle\big|^2(2\pi)^4\delta^{(4)}(k-p_m+p_n) \tag{3.23}$$

The domain of validity of $D^\pm$ is determined by requiring the convergence of the sum. For simplicity, we look at only the time argument

$$D^+(t) = \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\mathrm{e}^{i(E_n-E_m)t}\big|\langle n|\phi(0)|m\rangle\big|^2 \tag{3.24}$$

from which we read off $D^+(t)$ is defined on the strip $-\beta < \mathrm{Im}(t) < 0$ for a complex variable t.

Similarly, we get

$$D^-(x,y) = \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\mathrm{e}^{i(p_m-p_n)(x-y)}\big|\langle n|\phi(0)|m\rangle\big|^2 \tag{3.25}$$

$$D^-(t) = \frac{1}{Z}\sum_{n,m}\mathrm{e}^{-\beta E_n}\mathrm{e}^{i(E_m-E_n)t}\big|\langle n|\phi(0)|m\rangle\big|^2 \tag{3.26}$$

and find that $D^-(t)$ is defined within $0 < \mathrm{Im}(t) < \beta$.

There is a relation between $D^+(t)$ and $D^-(t)$

$$D^+(t-i\beta) = D^-(t) \tag{3.27}$$

which can be checked by comparing (3.24) and (3.26). This relation is just the KMS relation dressed in a different form.

In the Fourier frequency space, we have

$$D^+(k^0) = \int dt e^{ik^0 t} D^+(t) \tag{3.28}$$

$$D^-(k^0) = \int dt e^{ik^0 t} D^-(t) = \int dt e^{ik^0 t} D^+(t - i\beta) \tag{3.29}$$

Comparing these two, we obtain a relation between them

$$D^-(k^0) = e^{-\beta k^0} D^+ k^0 \tag{3.30}$$

We also define the *spectral density*

$$\rho(k^0) = D^+(k^0) - D^-(k^0) = \left(e^{\beta k^0} - 1\right) D^-(k^0) \tag{3.31}$$

with which we may rewrite the correlators as

$$D^+(k^0) = \frac{\rho(k^0)}{1 - e^{-\beta k^0}} = [1 + n(k^0)]\rho(k^0) \tag{3.32}$$

$$D^-(k^0) = \frac{\rho(k^0)}{e^{\beta k^0} - 1} = n(k^0)\rho(k^0) \tag{3.33}$$

where $n(E)$ is the Bose distribution factor

$$n(E) = \frac{1}{e^{\beta E} - 1} \tag{3.34}$$

We find for the spectral density

$$\begin{aligned}
\rho(k^0) &= \int_{-\infty}^{+\infty} dt e^{ik^0 t} \left[D^+(t) - D^-(t)\right] \\
&= \int_{-\infty}^{+\infty} dt e^{ik^0 t} \sum_{n,m} e^{-\beta E_n} \left|\langle n|\phi(0)|m\rangle\right|^2 \left[e^{i(E_n - E_m)t} - e^{-i(E_n - E_m)t}\right] \\
&= \sum_{n,m} e^{-\beta E_n} \left|\langle n|\phi(0)|m\rangle\right|^2 \left[\delta(k^0 - E_m + E_n) - \delta(k^0 + E_m - E_n)\right] \qquad (3.35)
\end{aligned}$$

from which we see that the spectral density is odd in k^0

$$\rho(-k^0) = -\rho(k^0) \tag{3.36}$$

Going back to the definition of $D^\pm$, we find

$$D^+(t) - D^-(t) = \langle [\phi(t), \phi(0)] \rangle_\beta \tag{3.37}$$

Differentiate with respect to t, we get from the LHS

$$\frac{d}{dt}\left(D^+(t) - D^-(t)\right) = \frac{d}{dt}\int \frac{dk^0}{2\pi}\mathrm{e}^{-ik^0t}\left(D^+(k^0) - D^-(k^0)\right)$$
$$= -i\int \frac{dk^0}{2\pi}k^0\mathrm{e}^{-ik^0t}\rho(k^0) \tag{3.38}$$

and from the RHS

$$\frac{d}{dt}\langle[\phi(t), \phi(0)]\rangle_\beta = -\langle[\phi(0), \pi(t)]\rangle_\beta \tag{3.39}$$

Recall that the equal time commutation relation is imposed as

$$[\phi(t), \pi(t)] = i \tag{3.40}$$

Therefore, differentiating both sides of (3.37) and taking the $t \to 0$ limit, we obtain

$$\int \frac{dk^0}{2\pi}k^0\rho(k^0) = 1 \tag{3.41}$$

showing that the spectral density is bounded for large k^0. As $k^0 \to \infty$, we shall expect $\rho(k^0) \to 1/(k^0)^2 \to 0$ at least.

For the free scalar fields we are dealing with, we can explicitly calculate the spectral density. We have

$$\rho_F(k^0) = \frac{1}{Z}\int dt\mathrm{e}^{ik^0t}\sum_n \mathrm{e}^{\beta E_n}\langle n|\phi(t)\phi(0) - \phi(0)\phi(t)|n\rangle$$
$$= \frac{2\pi}{2E_{\mathbf{k}}}\left(\delta(k^0 - E_{\mathbf{k}}) - \delta(k^0 + E_{\mathbf{k}})\right)$$
$$= 2\pi\epsilon(k^0)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right) \tag{3.42}$$

It is sometimes convenient to define the propagators in terms of imaginary time variables, so that we can work under the Euclidean space. This is usually referred to as the Matsubara propagator in literature. We define

$$\Delta(\tau) = \frac{1}{Z}\sum_n \langle n|\mathrm{e}^{\beta H}\phi(\tau)\phi(0)|n\rangle \tag{3.43}$$

whose Fourier transform is

$$\Delta(i\omega_n) = \int_0^\beta d\tau\mathrm{e}^{i\omega_n\tau}\Delta(\tau) \tag{3.44}$$

The Matsubara frequencies $\omega_n = 2\pi n/\beta$ can be read off from the condition

$$\Delta(\tau) = \frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n\tau}\Delta(i\omega_n) = \Delta(\tau + \beta) \tag{3.45}$$

The relation between imaginary and real time propagators can be found as

$$\Delta(\tau) = D^{+}(t = -i\tau) \tag{3.46}$$

$$= \int \frac{dk^0}{2\pi} \mathrm{e}^{-k^0\tau}[1 + n(k^0)]\rho(k^0) \tag{3.47}$$

So we can compute

$$\begin{aligned}
\Delta(i\omega_n) &= \int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} \int \frac{dk^0}{2\pi} \mathrm{e}^{-k^0\tau}[1 + n(k^0)]\rho(k^0) \\
&= \int \frac{dk^0}{2\pi} \int_0^\beta d\tau \mathrm{e}^{(i\omega_n - k^0)\tau}[1 + n(k^0)]\rho(k^0) \\
&= \int \frac{dk^0}{2\pi} \frac{\mathrm{e}^{(i\omega_n - k^0)\tau}}{i\omega_n - k^0}\Big|_0^\beta \left[1 + \frac{1}{\mathrm{e}^{\beta k^0} - 1}\right]\rho(k^0) \\
&= -\int \frac{dk^0}{2\pi} \frac{\rho(k^0)}{i\omega_n - k^0}
\end{aligned} \tag{3.48}$$

Substituting (3.42), we find the Matsubara propagator for free scalar particles

$$\Delta(i\omega_n) = -\frac{1}{2E_{\mathbf{k}}}\left(\frac{1}{i\omega_n - E_{\mathbf{k}}} - \frac{1}{i\omega_n + E_{\mathbf{k}}}\right) = \frac{1}{\omega_n^2 + E_{\mathbf{k}}^2} \tag{3.49}$$

Similarly, one can define the *retarded correlator* as

$$D_R(k^0) = \int dt \mathrm{e}^{ik^0 t}\theta(t)\left[D^{+}(t) - D^{-}(t)\right] \tag{3.50}$$

$$= i\int \frac{dk^{0\prime}}{2\pi} \frac{\rho(k^{0\prime})}{k^0 - k^{0\prime} + i\epsilon} \tag{3.51}$$

where we add a infinitesimal imaginary part $i\epsilon$ so that we close the contour and pick up poles with the correct signature in correspondence with our definition.

Comparing the retarded correlator with (3.48), we have

$$D_R(k^0) = -i\Delta(k^0 + i\epsilon) \tag{3.52}$$

The other useful correlator is the time-ordered *Feynman correlator*

$$D_F(t - t') = \langle \mathcal{T}\phi(t)\phi(t')\rangle_\beta \tag{3.53}$$

$$= \theta(t - t')D^{+}(t - t') + \theta(t' - t)D^{-}(t - t') \tag{3.54}$$

$\mathcal{T}$ means time ordering. We can compute in frequency space

$$\begin{aligned}
D_F(k^0) &= \int dt \mathrm{e}^{ik^0 t} D_F(t) \\
&= \int dt \mathrm{e}^{ik^0 t} \int \frac{dq^0}{2\pi}\frac{d\omega}{2\pi i}\left\{-\frac{\mathrm{e}^{-i\omega t}}{\omega - q^0 + i\epsilon}D^{+}(q^0) + \frac{\mathrm{e}^{-i\omega t}}{\omega - q^0 - i\epsilon}D^{-}(q^0)\right\} \\
&= \int \frac{dq^0}{2\pi}\frac{d\omega}{2\pi i}\left\{2\pi\delta(k^0 - \omega)\frac{iD^{+}(q^0)}{\omega - q^0 + i\epsilon} - 2\pi\delta(k^0 - \omega)\frac{iD^{-}(q^0)}{\omega - q^0 - i\epsilon}\right\} \\
&= \int \frac{dq^0}{2\pi}\left\{\frac{iD^{+}(q^0)}{k^0 - q^0 + i\epsilon} - \frac{iD^{-}(q^0)}{k^0 - q^0 - i\epsilon}\right\}
\end{aligned} \tag{3.55}$$

Using the spectral density function, we get

$$
\begin{aligned}
D_F(k^0) &= \int \frac{dq^0}{2\pi} \left\{ \frac{i\rho(q^0)}{k^0 - q^0 + i\epsilon} + in(q^0)\rho(q^0) \left[\frac{1}{k^0 - q^0 + i\epsilon} - \frac{1}{k^0 - q^0 - i\epsilon} \right] \right\} \\
&= \int \frac{dq^0}{2\pi} \frac{i\rho(q^0)}{k^0 - q^0 + i\epsilon} - in(k^0)\rho(k^0) \times \frac{\pi i}{2\pi} + in(k^0)\rho(k^0) \times \frac{-\pi i}{2\pi} \\
&= \int \frac{dq^0}{2\pi} \frac{i\rho(q^0)}{k^0 - q^0 + i\epsilon} + n(k^0)\rho(k^0)
\end{aligned}
\tag{3.56}
$$

At $T \to 0$, the Boson distribution factor $n(k^0) \to 0$, we recover the Feynman correlator in zero temperature field theory

$$
D_F(k^0; T = 0) = \int \frac{dq^0}{2\pi} \frac{i\rho(q^0)}{k^0 - q^0 + i\epsilon} \tag{3.57}
$$

To compute the Feynman correlator for free scalar fields, plug (3.42) into (3.56), we obtain

$$
\begin{aligned}
D_F(k^0) &= \int dq^0 \frac{i}{2E_{\mathbf{k}}} \frac{\delta(k^0 - E_{\mathbf{k}}) - \delta(k^0 + E_{\mathbf{k}})}{k^0 - q^0 + i\epsilon} + 2\pi\epsilon(k^0)n(k^0)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right) \\
&= \frac{i}{2E_{\mathbf{k}}} \frac{1}{k^0 - E_{\mathbf{k}} + i\epsilon} - \frac{i}{2E_{\mathbf{k}}} \frac{1}{k^0 + E_{\mathbf{k}} + i\epsilon} + 2\pi\epsilon(k^0)n(k^0)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right)
\end{aligned}
\tag{3.58}
$$

We want to write this in a neater form which looks more like the Feynman correlator in zero temperature field theories. Note that the second term

$$
\frac{-i}{2E_{\mathbf{k}}} \frac{1}{k^0 + E_{\mathbf{k}} + i\epsilon} = \frac{-i}{2E_{\mathbf{k}}} \frac{1}{k^0 + E_{\mathbf{k}} - i\epsilon} + \frac{-i}{2E_{\mathbf{k}}}(-2\pi i)\delta(k^0 + E_{\mathbf{k}}) \tag{3.59}
$$

Together with the first term, the first two terms of (3.58) give

$$
\begin{aligned}
&\frac{i}{2E_{\mathbf{k}}} \frac{1}{k^0 - E_{\mathbf{k}} + i\epsilon} - \frac{i}{2E_{\mathbf{k}}} \frac{1}{k^0 + E_{\mathbf{k}} - i\epsilon} - \frac{1}{2E_{\mathbf{k}}}(2\pi)\delta(k^0 + E_{\mathbf{k}}) \\
=&\frac{i}{(k^0)^2 - E_{\mathbf{k}}^2 + i\epsilon} - \frac{1}{2E_{\mathbf{k}}}(2\pi)\delta(k^0 + E_{\mathbf{k}})
\end{aligned}
\tag{3.60}
$$

From the third term we have

$$
2\pi\epsilon(k^0)n(k^0)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right) = \frac{2\pi}{2E_{\mathbf{k}}} n(E_{\mathbf{k}})\delta(k^0 - E_{\mathbf{k}}) - \frac{2\pi}{2E_{\mathbf{k}}} n(-E_{\mathbf{k}})\delta(k^0 + E_{\mathbf{k}}) \tag{3.61}
$$

Noting the property of the distribution factor

$$
n(-E) = \frac{1}{\mathrm{e}^{-\beta E} - 1} = -1 - \frac{1}{\mathrm{e}^{\beta E} - 1} = -1 - n(E) \tag{3.62}
$$

The third term of (3.58) gives rise to

$$
\begin{aligned}
&\frac{2\pi}{2E_{\mathbf{k}}} n(E_{\mathbf{k}})\delta(k^0 - E_{\mathbf{k}}) + \frac{2\pi}{2E_{\mathbf{k}}}\delta(k^0 + E_{\mathbf{k}}) + \frac{2\pi}{2E_{\mathbf{k}}} n(E_{\mathbf{k}})\delta(k^0 + E_{\mathbf{k}}) \\
=&2\pi n(|k^0|)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right) + \frac{2\pi}{2E_{\mathbf{k}}}\delta(k^0 + E_{\mathbf{k}})
\end{aligned}
\tag{3.63}
$$

Putting together all these terms in (3.58), we obtain the Feynman propagator for free scalar particles

$$D_F(k^0) = \frac{1}{k^0 + E_{\mathbf{k}} - i\epsilon} + 2\pi n(|k^0|)\delta\left((k^0)^2 - E_{\mathbf{k}}^2\right) \tag{3.64}$$

3.2 Interactions and perturbative theory

3.2.1 Feynman rules

Let's now add into the scalar fields a ϕ^4 interaction.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 = \mathcal{L}_0 + \mathcal{L}_I \tag{3.65}$$

The full Lagrangian breaks into a free part and an interaction part. The coupling constant λ is dimensionless and assumed to be small, so we can study the contributions from the interaction term perturbatively in terms of λ.

The partition function reads

$$Z = \int \mathcal{D}\phi \mathrm{e}^S = \int \mathcal{D}\phi \mathrm{e}^{S_0+S_I} = \int \mathcal{D}\phi \mathrm{e}^{S_0}(1 + S_I + \frac{1}{2}S_I^2 + \cdots) \tag{3.66}$$

We also break the partition function into a free piece and an interaction piece, and expand around the free partition function. Consider the logarithm of the partition function of interest

$$\begin{aligned}\ln Z &= \ln(Z_0 + Z_I)\\ &= \ln\left(\int \mathcal{D}\phi \mathrm{e}^{S_0} + \mathrm{e}^{S_0}\sum_{n=1}^{+\infty}\frac{S_I^n}{n!}\right)\\ &= \ln Z_0 + \frac{Z_I}{Z_0} - \frac{Z_I^2}{2Z_0^2} + \cdots\end{aligned} \tag{3.67}$$

The first order perturbation expansion in λ originates only from the term Z_I/Z_0. More specifically, this is

$$(\ln Z)_1 = \frac{\int \mathcal{D}\phi S_I \mathrm{e}^{S_0}}{\int \mathcal{D}\phi \mathrm{e}^{S_0}} \tag{3.68}$$

Substitute the field expansion (3.9), one obtains

$$\begin{aligned}(\ln Z)_1 =& \frac{1}{Z_0}\int \mathcal{D}\phi_{n,\mathbf{k}}(-\lambda)\int_0^\beta d\tau \int d^3\mathbf{x}\frac{\beta^2}{V^2}\sum_{n_1,\cdots,n_4}\sum_{\mathbf{p}_1,\cdots,\mathbf{p}_4}\phi_{n_1,\mathbf{p}_1}\cdots\phi_{n_4,\mathbf{p}_4}\\ &\times \mathrm{e}^{i(\omega_{n_1}+\cdots+\omega_{n_4})\tau}\mathrm{e}^{i(\mathbf{p}_1+\cdots+\mathbf{p}_4)\mathbf{x}}\mathrm{e}^{-\frac{\beta^2}{2}\sum_{n,\mathbf{k}}|\phi_{n,\mathbf{k}}|^2(\omega_n^2+\mathbf{k}^2+m^2)}\end{aligned} \tag{3.69}$$

where the denominator is

$$Z_0 = \int \mathcal{D}\phi_{n,\mathbf{p}}\mathrm{e}^{-\frac{\beta^2}{2}\sum_{n,\mathbf{k}}|\phi_{n,\mathbf{p}}|^2(\omega_n^2+\mathbf{p}^2+m^2)} \tag{3.70}$$

The integrations $\int_0^\beta d\tau$ and $\int d^3\mathbf{x}$ will give rise to a factor of

$$\beta V \delta(\omega_{n_1} + \cdots + \omega_{n_4}) \cdot \delta^{(3)}(\mathbf{p}_1 + \cdots + \mathbf{p}_4) \tag{3.71}$$

The only non-zero contributions are from $\omega_{n_1} = -\omega_{n_2}$, $\mathbf{p}_1 = -\mathbf{p}_2$ and $\omega_{n_3} = -\omega_{n_4}$, $\mathbf{p}_3 = -\mathbf{p}_4$ and two other possible permutations. Thus

$$\begin{aligned}(\ln Z)_1 &= 3\frac{1}{Z_0}(-\lambda)\beta V \prod_{n,\mathbf{k}} \int \mathcal{D}\phi_{n,\mathbf{k}} \frac{\beta^2}{V^2} \sum_{l,m} \sum_{\mathbf{p},\mathbf{q}} |\phi_{l,\mathbf{p}}|^2 |\phi_{m,\mathbf{q}}|^2 \mathrm{e}^{-\frac{\beta^2}{2}\phi^2_{n,\mathbf{k}}(\omega_n^2 + \mathbf{k}^2 + m^2)} \\ &= -3\lambda\beta V \left[\frac{V}{\beta} \sum_{n,\mathbf{p}} \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}\right]^2\end{aligned} \tag{3.72}$$

Taking the continuum limit, we yield

$$(\ln Z)_1 = -3\lambda\beta V \left[\frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}\right]^2 \tag{3.73}$$

The term in the square bracket is exactly the free scalar propagator in imaginary time (3.49). Diagrammatically, we can represent the first order corrections to the partition function in terms of a Feynman diagram

$$(\ln Z)_1 = -3 \times (\beta V) \times \bigcirc\!\!\bigcirc \tag{3.74}$$

We can also examine the second order corrections in λ. The contributions come from two pieces in (3.67), Z_I/Z_0 and $-Z_I^2/2Z_0^2$. From $-Z_I^2/2Z_0^2$, we get

$$-\frac{1}{2}\left[\frac{\mathcal{D}\phi S_I \mathrm{e}^{S_0}}{\mathcal{D}\mathrm{e}^{S_0}}\right]^2 = -\frac{1}{2}\left[3\beta V \bigcirc\!\!\bigcirc\right]^2 \tag{3.75}$$

From Z_I/Z_0, we pick out the contributions of the second order

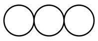
$$\frac{1}{Z_0}\int \mathcal{D}\phi \mathrm{e}^{S_0} \frac{S_I^2}{2} = \frac{1}{Z_0}\int \mathcal{D}\phi \mathrm{e}^{S_0} \frac{1}{2}\left[-\lambda \int d^4x \phi^4\right]^2 \tag{3.76}$$

There are several topologically distinct Feynman diagrams corresponding to this expression. Ignoring the factor of one half and some multiplicative factors relating to the symmetry of the diagrams, we draw these diagrams

(3.77)

The multiplicative factors can be worked out by starting from two separate vertices with 4 legs and counting the total number of possible ways to connect different legs to form the consequential diagram. For the first diagram, we have 3×3 ways to complete this. Picking up the factor of one half from the Taylor expansion, we see that the first diagram cancels out the contribution of (3.75). To obtain the second diagram of (3.77), we choose two legs from one vertex to connect with two legs from the other vertex, of which there are $4 \times 3/2 = 6$ possible choices for each vertex, and thus $6 \times 6 \times 2 = 72$ ways in total. The third diagram can be done in $4 \times 3 \times 2$ ways.

Therefore, the second order corrections to the partition function looks like

$(\ln Z)_2 = 36(\beta V)^2$  $+ 12(\beta V)^2$  (3.78)

The disconnected pieces from the numerator and the denominator are canceled out. As we should point out, this is true at all orders of the perturbation expansions, though we only demonstrated this cancellation to the second order. This could be understood in this way. If there exists a contribution from n pieces of disconnected diagrams, then each of the disconnected piece will contribute a factor of V and lead to an overall factor of V^n. But the logarithm of the partition function is proportional to the free energy, an extensive quantity, and therefore must be proportional to V. The number of the disconnected pieces must be one only, i.e., the contributing diagram must be connected.

We can sum up our discussions to write down the Feynman rules for the ϕ^4-interacting scalar field theories.

1. Draw all topologically inequivalent connected Feynman diagrams.
2. Assign a factor of $\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\Delta(i\omega_n, \mathbf{p})$ to each line.
3. Assign a factor of $(-\lambda)$ to each vertex.
4. Include a factor of $\frac{(2\pi)^3}{\beta}\delta(\omega_{in} - \omega_{out})\delta^{(3)}(\mathbf{p}_{in} - \mathbf{p}_{out})$ at each vertex due to energy-momentum conservation.
5. Determine the overall combinatoric symmetry factor.
6. There will be an overall factor of $\beta(2\pi)^3\delta(0) = \beta V$ left over.

3.2.2 Progapators and self-energies

The existence of interactions will modify the propagators. A finite temperature propagator in coordinate space is defined by

$$D(\tau, \mathbf{x}) = \langle \phi(\tau, \mathbf{x})\phi(0, 0)\rangle_\beta = \frac{1}{Z}\int \mathcal{D}\phi\phi(\tau, \mathbf{x})\phi(0, 0)e^{\int_0^\beta d\tau \int d^3\mathbf{x}\mathcal{L}} \tag{3.79}$$

Transform into the frequency-momentum space

$$D(\omega_n, \mathbf{p}) = \frac{1}{Z}\int_0^\beta d\tau \int d^3\mathbf{x}e^{-i(\omega_n\tau + \mathbf{p}\cdot\mathbf{x})}D(\tau, \mathbf{x}) \tag{3.80}$$

$$= \frac{\beta}{V}\sum_{m,\mathbf{q}}\sum_{m',\mathbf{q}'}\langle \phi_{m,\mathbf{q}}\phi_{m',\mathbf{q}'}\rangle_\beta \int_0^\beta d\tau \int d^3\mathbf{x}e^{i(\omega_m - \omega_n)\tau}e^{i(\mathbf{q}-\mathbf{p})\mathbf{x}} \tag{3.81}$$

The ensemble average $\langle \phi_{m,\mathbf{q}} \phi_{m',\mathbf{q}'} \rangle_\beta$ is zero by symmetric integration unless $m = -m'$ and $\mathbf{q} = -\mathbf{q}'$. So we have

$$D(\omega_n, \mathbf{p}) = \beta^2 \langle \phi_{n,\mathbf{p}} \phi_{-n,-\mathbf{p}} \rangle_\beta \tag{3.82}$$

Write this in terms of a path integral,

$$D(\omega_n, \mathbf{p}) = \beta^2 \frac{\int \mathcal{D}\phi \phi_{n,\mathbf{p}} \phi_{-n,-\mathbf{p}} \mathrm{e}^S}{\int \mathcal{D}\phi \mathrm{e}^S} \tag{3.83}$$

from which we may find

$$-2 \frac{\delta \ln Z}{\delta D_0^{-1}} = \frac{-2}{Z} \frac{\delta Z}{\delta D_0^{-1}} = \frac{\beta^2}{Z} \int \mathcal{D}\phi \phi_{n,\mathbf{p}} \phi_{-n,-\mathbf{p}} \mathrm{e}^S = D(\omega_n, \mathbf{p}) \tag{3.84}$$

$D(\omega_n, \mathbf{p})$ can be nicely expressed as a functional derivative of $\ln Z$ with respect to free propagator $D_0(\omega_n, \mathbf{p})$. Further we have

$$D(\omega_n, \mathbf{p}) = -2 \frac{\delta \ln Z}{\delta D_0^{-1}} = 2 D_0^2 \frac{\delta \ln Z}{\delta D_0} \tag{3.85}$$

We define the *self energy* for scalar fields

$$\Pi(\omega_n, \mathbf{p}) = D^{-1}(\omega_n, \mathbf{p}) - D_0^{-1}(\omega_n, \mathbf{p}) \tag{3.86}$$

Noting that

$$\frac{\delta \ln Z_0}{\delta D_0} = \frac{1}{2} \frac{\delta}{\delta D_0} \sum_{n,\mathbf{p}} \ln \left[\beta^{-2} D_0(\omega_n, \mathbf{p}) \right] = \frac{1}{2} D_0^{-1} \tag{3.87}$$

we can compute from (3.84)

$$D(\omega_n, \mathbf{p}) = 2 D_0^2 \left\{ \frac{\delta \ln Z_0}{\delta D_0} + \frac{\delta \ln Z_1}{\delta D_0} + \cdots \right\} = D_0 + 2 D_0^2 \frac{\delta \ln Z_1}{\delta D_0} + \cdots \tag{3.88}$$

where we expand $\ln Z$ into a perturbative series. We also expand the self energy

$$\Pi(\omega_n, \mathbf{p}) = \Pi_1(\omega_n, \mathbf{p}) + \Pi_2(\omega_n, \mathbf{p}) + \cdots \tag{3.89}$$

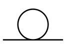
Solve for (3.86) to the first order, we obtain

$$1 - D_0 \Pi_1 = 1 + 2 D_0 \frac{\delta \ln Z_1}{\delta D_0} \tag{3.90}$$

or

$$\Pi_1 = -2 \frac{\delta \ln Z_1}{\delta D_0} \tag{3.91}$$

Recall that we interpreted the partition functions $\ln Z$with Feynman diagrams. Diagrammatically, differentiating $\ln Z$ with respect to the free propagator D_0 is equivalent to cutting lines in the diagrams. We get

$$\Pi_1 = -2 \frac{\delta}{\delta D_0} (-3 \times (\beta V) \times$$  $$) = -12$$  (3.92)

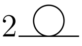
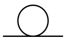
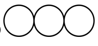
There is a factor of 2 because there are 2 different ways to cut lines in the figure-eight diagram for $\ln Z_1$.

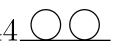
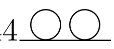
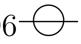
At the second order, we yield

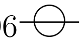
$$-D_0\Pi_2 + D_0\Pi_1 D_0\Pi_1 = 2D_0\frac{\delta \ln Z_2}{\delta D_0} \tag{3.93}$$

or

$$\Pi_2 = \Pi_1 D_0 \Pi_1 - 2\frac{\delta \ln Z_2}{\delta D_0}$$

$= (-12$  $) \times (-12$  $) - 2\dfrac{\delta}{\delta D_0}(36$  $+ 12$  $)$

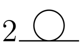
$= 144$  $- 144$  $- 144$  $- 96$ 

$= -144$  $- 96$  (3.94)

We may obtain the expressions for higher order corrections with the same methods. In principle, the self energies can be computed to arbitrary high orders. However, we will see later that this naive perturbative expansion is poorly convergent due to the infrared divergences.

3.2.3 Thermal mass and phase transitions

We have obtained a diagrammatic representation for the self energies. These diagrams can be evaluated by using the Feynman rules. Let's evaluate the self energy at the lowest order corresponding to the one-loop diagram. Recalling the Feynman rules, we write

$\Pi_1 = -12$ 

$$\begin{aligned}
&= -12(-\lambda)\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2} \\
&= 12\lambda \int_C \frac{dp^0}{2\pi i}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{(p^0)^2 - \omega_p^2}\frac{1}{2}\coth\frac{\beta p^0}{2} \\
&= 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_p}\frac{1}{2}\coth\frac{\beta\omega_p}{2} \times 2 \\
&= 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_p}\left(1 + \frac{2}{\mathrm{e}^{\beta\omega_p} - 1}\right)
\end{aligned} \tag{3.95}$$

Like the partition function of a free scalar field at finite temperature we calculated before, the self energy also splits into two parts.

The temperature independent part

$$\Pi_{vac} = 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_p} = 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{(p^0)^2 - \omega_p^2} \tag{3.96}$$

is just the vacuum one-loop self energy calculated in many quantum field theory textbooks with a Wick rotation of p^0 to imaginary axis. With $p^0 = ip^4$ we rewrite

$$\begin{aligned}\Pi_{vac} &= 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp^4}{2\pi i} \frac{-i}{-(p^4)^2 - \omega_p^2} \\ &= 12\lambda \int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{1}{(p^4)^2 + \omega_p^2} \end{aligned} \tag{3.97}$$

The integral is unfortunately divergent. We should set a large but finite cut-off momentum Λ_c. When $m = 0$, the integration gives

$$12\lambda \cdot \frac{2\pi^2}{16\pi^4} \frac{p^2}{2}\bigg|_0^{\Lambda_c} = \frac{3\lambda}{4\pi^2}\Lambda_c^2 \tag{3.98}$$

while $m \neq 0$, the result becomes

$$12\lambda \cdot \frac{2\pi^2}{16\pi^4} \frac{1}{2}\left[p^2 - m^2 \ln\frac{p^2 + m^2}{m^2}\right]\bigg|_0^{\Lambda_c} = \frac{3\lambda}{4\pi^2}\left(\Lambda_c^2 - m^2 \ln\frac{\Lambda_c^2}{m^2}\right) \tag{3.99}$$

However, under a proper counter term renormalization scheme, $m^2 = m_0^2 + \delta m^2$, where m_0 is the bare mass and δm shall be treated as a counter term, we can always choose δm^2 such that the vacuum self energy vanishes.

$$\Pi_{vac}^{ren} \equiv 0 \tag{3.100}$$

We shall emphasize that renormalization at zero temperature is sufficient to make the full theory well-defined in non-zero temperature cases, because the thermal contributive part is suppressed at high energies due to the property of distribution function, which contains an exponential factor $\mathrm{e}^{-\beta E}$.

We are then left with a thermal part

$$\Pi_T = 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega_p} \frac{1}{\mathrm{e}^{\beta\omega_p} - 1} \tag{3.101}$$

The integral can be easily evaluated in the massless limit $m = 0$

$$\begin{aligned}\Pi_T &= 12\lambda \cdot 4\pi \int \frac{dpp^2}{(2\pi)^3} \frac{1}{p} \frac{1}{\mathrm{e}^{\beta p} - 1} \\ &= \frac{12\lambda}{4\pi^2} \frac{1}{\beta^2} \int dx \frac{x}{\mathrm{e}^x - 1} \\ &= \lambda T^2 \end{aligned} \tag{3.102}$$

where we used the result for the dimensionless integral

$$\int_0^\infty dx \frac{x}{\mathrm{e}^x - 1} = \frac{\pi^2}{6} \tag{3.103}$$

The thermal background introduces a mass for scalar particles of order λT^2. The appearance of the thermal mass has an important result, and we make several comments on it. If we take a scalar Lagrangian with a negative square mass

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4 = \mathcal{L}_0 + \mathcal{L} + I \tag{3.104}$$

At zero temperature, the effective potential takes its minimum at non-zero values of the field $\phi_c = \sqrt{\mu/2\lambda}$. Hence the symmetry is spontaneously broken. The vacuum mass will continue to dominate at low temperatures. But owing to the positive square-mass generated by a thermal background, the effective potential will behave differently provided we keep increasing the temperature. When the magnitude of the thermal mass is larger than that of the vacuum mass, the minimum of the effective potential is acquired when we take all the field arguments to be zero $\phi_c = 0$. We should expect the broken symmetry to be restored at some critical temperature. There must exist a *phase transition* between the high and low temperature domain, which is a significant consequence for our understanding of the early universe. The broken symmetry of the world we are living in, which is a low-temperature system with $T \approx 3K$, shall be restored when tracing back to early times. We know that the Higgs mechanism of spontaneous symmetry breaking is responsible for the particle masses, so we might infer that all particles were massless in the early universe. Although the details of the phase transition should be discussed more carefully, since we have only studied the mass corrections to the first order. However, our calculation is sufficient to convey us a general idea of how temperatures could influence the behaviors of our scalar field model.

3.2.4 Partition function

We want to study the corrections to the partition function to the first order

$$\begin{aligned}
(\ln Z)_1 &= -3\beta V \bigcirc\!\!\bigcirc \\
&= -3\lambda\beta V \left(\frac{1}{\beta}\sum_n \int d^3\mathbf{p}(2\pi)^3 \frac{1}{(p^0)^2 - \omega_p^2}\right)^2 \\
&\to -3\lambda\beta V \left(\int d^3p(2\pi)^3 \frac{1}{2p}\frac{2}{\mathrm{e}^{\beta p} - 1}\right)^2 \\
&= -3\lambda\beta V \left(\frac{1}{2\pi^2}\frac{1}{\beta^2}\frac{\pi^2}{6}\right)^2 \\
&= -\frac{\lambda V}{48\beta^3}
\end{aligned} \tag{3.105}$$

where in the third step, we again take the massless limit. We also drop the vacuum part as long as the vacuum pressure is renormalized to zero.

The corresponding first order correction to the pressure is

$$P_1 = \frac{(\ln Z)_1}{\beta V} = -\frac{\lambda}{48} T^4 \tag{3.106}$$

Together with the free field pressure (3.19), the pressure of interacting fields looks like

$$P = T^4 \left(\frac{\pi^2}{90} - \frac{\lambda}{48} + \cdots \right) \tag{3.107}$$

As we will see soon, the next to leading order corrections are not of order λ^2 as one would expect, but of order $\lambda^{3/2}$. The naive perturbation method is problematic if one studies the thermal field theories more carefully. We will discuss these unexpected behaviors in Section 5.

4 Gauge Theories at Finite Temperature

Up till now, we have been studying the scalar theories only. However, the world we are living in is consisted of spin-half fermions and gauge bosons, which are well explained by gauge theories. Quantum electrodynamics (QED) and quantum chromodynamics (QCD), being the two most important and successful gauge theories, describe the mechanism behind almost all phenomena to a very high accuracy. In this section, we will study the generalization of gauge theories to finite temperatures in details.

4.1 Fermions

4.1.1 Partition function

Recall the Lagrangian for free electrons is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{4.1}$$

The field operators ψ and $\bar{\psi}$ should be treated as independent variables. The equation of motion is the famous Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{4.2}$$

The conjugate momentum of the field ψ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = i\psi^\dagger \tag{4.3}$$

We can expand the fields in terms of creation and annihilation operators

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left\{b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})\mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}\right\} \tag{4.4}$$

where $u_s(\mathbf{p})$, $v_s(\mathbf{p})$ are the free solutions to the Dirac equation with positive and negative frequencies, respectively, and the lower index s denotes for two different spin states. Similar expansion holds for $\bar{\psi}(x)$.

The equal time anti-commutation relation is imposed as

$$\{\psi_a(t,\mathbf{x}), \pi_b(t,\mathbf{y})\} = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab} \tag{4.5}$$

or equivalently

$$\{b_r(\mathbf{p}), b_s^\dagger(\mathbf{q})\} = \{d_r(\mathbf{p}), d_s^\dagger(\mathbf{q})\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{rs} \tag{4.6}$$

The anti-commutators reveal the fermionic nature of electrons. Unlike scalar particles, the Hilbert space of a system of electrons is finite dimensional. Consider a single mode of the system of electrons, then the basis of the Hilbert space is conventionally given by two states $|0\rangle$ and $|1\rangle$, with an occupation number of 0 and 1 respectively. The trace that we need to take when computing the partition function for this single mode becomes a sum over only two states.

$$Z_p = \sum_{n=0}^{1} \langle n|\mathrm{e}^{-\beta H}|n\rangle = \langle 0|0\rangle + \mathrm{e}^{-\beta E_{\mathbf{p}}}\langle 1|1\rangle = 1 + \mathrm{e}^{-\beta E_{\mathbf{p}}} \tag{4.7}$$

We sum over all the modes to get the full partition function

$$\ln Z = \prod_p (1 + \mathrm{e}^{-\beta E_{\mathbf{p}}}) = 4V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln(1 + \mathrm{e}^{-\beta E_{\mathbf{p}}}) \tag{4.8}$$

The appearance of the factor of 4 in this expression for fermions is due to the degrees of freedom. There is a factor of 2 responsible for the existence of two sets of creation operators which generate anti-antiparticles as well as particles, and another factor of 2 because the fermions we deal with are of spin one-half and thus have 2 different spin states.

Since path integral turns out to be a powerful method to deal with gauge theories, hence we expect to construct a path integral representation for fermionic fields, so that we can include gauge interactions and study them in a similar approach later.

We define the *fermionic coherent states* $|\eta\rangle$ by

$$|\eta\rangle = \mathrm{e}^{-\eta a^\dagger}|0\rangle = (1 - \eta a^\dagger)|0\rangle = |0\rangle - \eta|1\rangle \tag{4.9}$$

The conjugate coherent states $\langle\eta|$ are given by

$$\langle\eta| = \langle 0|\mathrm{e}^{-a\eta^*} = \langle 0|(1 - a\eta^*) = \langle 0| - \langle 1|\eta^* \tag{4.10}$$

The numbers η, η^* are the basis of the Grassmann algebra.

A useful identity of Grassmannian variables is

$$\int d\eta^* d\eta \mathrm{e}^{-\eta^* a \eta} = \int d\eta^* d\eta (1 - \eta^* a \eta) = a \tag{4.11}$$

Some other useful identities are

$$1 = \int d\eta^* d\eta \mathrm{e}^{-\eta^*\eta} |\eta\rangle\langle\eta| \tag{4.12}$$

$$\mathrm{Tr} A = \int d\eta^* d\eta \mathrm{e}^{-\eta^*\eta} \langle -\eta|A|\eta\rangle \tag{4.13}$$

They can be easily checked by expanding the exponential terms and using the defining properties of the Grassmann algebra.

With the help of (4.13), we are ready to write down the fermion partition function in the form of path integrals

$$Z = \mathrm{Tr} e^{-\beta H} = \int d\eta^* d\eta e^{-\eta^*\eta} \langle -\eta | e^{-\beta H} | \eta \rangle \tag{4.14}$$

By splitting β into a large number of small intervals and inserting a set of unity identities (4.12), after some work one obtains

$$Z = \int \mathcal{D}\eta^* \mathcal{D}\eta \exp \left\{ - \int_0^\beta d\tau \left[\eta^* \frac{\partial \eta}{\partial \tau} + H(\eta^*, \eta) \right] \right\} \tag{4.15}$$

with a suitable choice of boundary condition

$$\eta(\beta) = -\eta(0) \tag{4.16}$$

The partition function for the Dirac fermion fields takes the form

$$\int \mathcal{D}\psi^* \mathcal{D}\psi \exp \left\{ \int_0^\beta d\tau d^3\mathbf{x} \left[-\psi^*(\tau, \mathbf{x}) \frac{\partial \psi(\tau, \mathbf{x})}{\partial \tau} - \psi^*(\tau, \mathbf{x})(-i\gamma^0\gamma^i\partial_i + m\gamma^0)\psi(\tau, \mathbf{x}) \right] \right\} \tag{4.17}$$

Going back to real time variable $t = -i\tau$, we find

$$-\psi^* \frac{\partial \psi}{\partial \tau} = i\psi^* \frac{\partial \psi}{\partial t} = i\bar{\psi}\gamma^0\partial_0\psi \tag{4.18}$$

$$i\psi^*\gamma^0\gamma^i\partial_i\psi = i\bar{\psi}\gamma^i\partial_i\psi \tag{4.19}$$

Therefore the path integral representation for the partition function is

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ i \int_0^{-i\beta} dt d^3\mathbf{x} \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \right\} \tag{4.20}$$

The exponential term is just the Lagrangian (4.1). Hence the expression is further simplified to be

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ i \int_0^{-i\beta} dt d^3\mathbf{x} \mathcal{L}(t) \right\} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ - \int_0^\beta d\tau d^3\mathbf{x} \mathcal{L}(\tau) \right\} \tag{4.21}$$

which is analogous to (2.19) but the functional integration variables are now anti-commuting objects.

It is convenient to deal with the fermion fields by expanding them into the frequency-momentum space, but note that we have the constraint $\psi(\tau = \beta) = -\psi(\tau = 0)$, which can only be satisfied by a discrete set of frequencies. The Fourier expansion is

$$\psi(\mathbf{x}, \tau) = \frac{1}{\sqrt{V}} \sum_{n,\mathbf{p}} e^{i(\omega_n\tau + \mathbf{p}\cdot\mathbf{x})} \psi(\omega_n, \mathbf{p}) \tag{4.22}$$

where the allowed Matsubara frequencies are

$$\omega_n = \frac{(2n+1)\pi}{\beta} \tag{4.23}$$

which is consistent with our discussion (2.16)

Insert (4.22) into the action $S = \int_0^{-i\beta} dt d^3\mathbf{x}\mathcal{L} = \int_0^{-i\beta} dt d^3\mathbf{x}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$, we get

$$S = -\beta \sum_{\mathbf{p}} \bar{\psi}^*_{n,\mathbf{p}}(i\omega_n + \gamma^0\gamma^i p^i + m\gamma^0)\psi_{n,\mathbf{p}} \tag{4.24}$$

Note that in the integrand, the field arguments ψ^* and ψ are 4-component spinors, and the term in the round bracket containing γ-matrices has a 4×4 matrix structure. In order to carry out this integral, we shall use a generalized version of (4.11)

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-\bar{\psi}M\psi) = \det M \tag{4.25}$$

Integrating over $\mathcal{D}\psi^*$ and $\mathcal{D}\psi$, we get the determinant of the matrix element

$$\beta(i\omega_n + \gamma^0\gamma^i p^i + m\gamma^0) \tag{4.26}$$

Using the Dirac representation for the γ-matrices, this matrix can be written explicitly as

$$\beta \begin{pmatrix} i\omega_n - \mathbf{p} & 0 & m & 0 \\ 0 & i\omega_n + \mathbf{p} & 0 & m \\ m & 0 & i\omega_n + \mathbf{p} & 0 \\ 0 & m & 0 & i\omega_n - \mathbf{p} \end{pmatrix} \tag{4.27}$$

The determinant of this matrix is worked out to be

$$\det(\cdots) = \beta^4 \left[(i\omega_n)^2 - \mathbf{p}^2 - m^2\right]^2 = \beta^4(\omega_n + E_{\mathbf{p}}^2)^2 \tag{4.28}$$

So up to an unimportant multiplicative constant independent of temperature, the partition function is

$$\ln Z = \sum_{n,\mathbf{p}} \ln\left[\beta^4(\omega_n + E_{\mathbf{p}}^2)^2\right] \tag{4.29}$$

In the continuum momentum limit

$$\ln Z = 2V \sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln\left[\beta^2(\omega_n + E_{\mathbf{p}}^2)\right] \tag{4.30}$$

The frequency sum can be performed with a similar trick using a contour integral as before. We first write the logarithm term in the integrand as

$$\ln\left[\beta^2(\omega_n + E_{\mathbf{p}}^2)\right] = \int_1^{\beta^2 E_{\mathbf{p}}^2} \frac{d\theta^2}{\beta^2\omega_n^2 + \theta^2} \tag{4.31}$$

and then interpret the sum as the residuals of a clever choice of contour integral by using the hyperbolic tangent function. We simply give the result of the frequency sum here, the reader can refer to the appendix for details. From (A.9), we have

$$\sum_n \frac{1}{\beta^2\omega_n^2+\theta^2} = \frac{1}{2\theta} - \frac{1}{\theta}\frac{1}{\mathrm{e}^\theta+1} \tag{4.32}$$

The last term is related to the vacuum contribution to the partition function, of which we are not interested and thus will be dropped from now on. The auxiliary integration over $d\theta$ reads

$$\begin{aligned}\int_1^{\beta^2E_\mathbf{p}^2} d\theta^2\left(\frac{1}{2\theta}-\frac{1}{\theta}\frac{1}{\mathrm{e}^\theta+1}\right) &= \int_1^{\beta E_\mathbf{p}} d\theta\left(1-\frac{2}{\mathrm{e}^\theta+1}\right)\\ &= \beta E_\mathbf{p} + 2\ln(1+\mathrm{e}^{-\beta E_\mathbf{p}}) - 2\ln(1+\mathrm{e}^{-1}) \end{aligned} \tag{4.33}$$

where the last temperature independent constant term can be neglected.

Therefore we arrive at the final expression of the partition function for pure fermions

$$\ln Z = 4V\int\frac{d^3\mathbf{p}}{(2\pi)^3}\left[\frac{1}{2}\beta E_\mathbf{p} + \ln(1+\mathrm{e}^{-\beta E_\mathbf{p}})\right] \tag{4.34}$$

The first term again owes its appearance to the zero-point energy. This expression is obviously consistent with (4.8), which we computed in a different approach.

We also notice that this expression is very similar to the analogous formula (3.17) for scalar particles, with two major differences. (i) The change of signs, arising from difference between the fermionic and bosonic nature of the particles we are studying. For fermions, the integration over anti-commuting Grassmannian variables instead of ordinary commuting c-numbers gives an overall minus sign, and the difference in periodicity conditions leads to another change of sign for the term in the logarithm. (ii) The appearance of the factor of 4, which is absent for real scalar particles, arising from the fact that electrons have 4 degrees of freedom.

4.1.2 Electron propagators

The studies of electron propagators are very similar to our discussions on scalar propagators. Since the methods we use are almost basically the same, we are not going into too much details for electron propagators. The reader can compare the relevant discussions in the Section 3.1 for a reference.

As for scalar fields, we can define two-point correlators for electrons

$$S^+_{\alpha\beta}(x,y) = \langle\psi_\alpha(x)\bar\psi_\beta(y)\rangle_\beta \tag{4.35}$$

$$S^-_{\alpha\beta}(x,y) = -\langle\bar\psi_\beta(y)\psi_\alpha(x)\rangle_\beta = -S^+_{\alpha\beta}(y,x) \tag{4.36}$$

Note the minus sign in the definition for $S^-_{\alpha\beta}(x,y)$, which is necessary to derive the correct KMS relation for fermions

$$S^+(t - i\beta) = -S^-(t) \tag{4.37}$$

One may find a relation similar to (3.30) as

$$S^-_{\alpha\beta}(p^0) = -\mathrm{e}^{-\beta p^0} S^+_{\alpha\beta}(p^0) \tag{4.38}$$

We also define the *spectral density* function

$$\rho_{\alpha\beta}(p^0) = S^+_{\alpha\beta}(p^0) - S^-_{\alpha\beta}(p^0) \tag{4.39}$$

with which we rewrite the correlators as

$$S^+_{\alpha\beta}(p^0) = [1 - \tilde{n}(p^0)]\rho_{\alpha\beta}(p^0) \tag{4.40}$$

$$S^-_{\alpha\beta}(p^0) = -\tilde{n}(p^0)\rho_{\alpha\beta}(p^0) \tag{4.41}$$

where $\tilde{n}(E)$ is the Fermi-Dirac distribution factor

$$\tilde{n}(E) = \frac{1}{\mathrm{e}^{\beta E} + 1} \tag{4.42}$$

With the same methods in our treatments of scalar propagators, one finds the electron Matsubara propagator

$$S(i\omega_n, \mathbf{p}) = -\int \frac{dp^0}{2\pi} \frac{\rho(p^0)}{i\omega_n - p^0} \tag{4.43}$$

which is an analogy to (3.48).

The spectral density for free electron fields is

$$\rho_{\alpha\beta}(p^0) = 2\pi\epsilon(p^0)\delta\big((p^0)^2 - E^2_{\mathbf{p}}\big)(\not{p} + m)_{\alpha\beta} \tag{4.44}$$

from which we yield the free Matsubara propagator for electrons

$$S(i\omega_n, \mathbf{p}) = -\frac{\not{p} - m}{\omega_n^2 + E^2_{\mathbf{p}}} = -(\not{p} - m)\tilde{\Delta}(i\omega_n, \mathbf{p}) \tag{4.45}$$

where for the convenience of writing in the future, we defined a Euclidean propagator for fermions

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + E^2_{\mathbf{p}}} \tag{4.46}$$

4.2 Quantum electrodynamics

4.2.1 QED Lagrangian

The free Dirac action (4.1) is invariant under a global phase transformation

$$\psi(x) \to \mathrm{e}^{i\alpha}\psi(x) \tag{4.47}$$

with a fixed phase parameter α. Nevertheless, the theory is no longer invariant provided that α has a dependence on local space-time coordinate x. To retain the invariance under the so-called local phase transformation, we need include gauge potentials A_μ into our theory. The gauge field together with the original fermion field will leave the Lagrangian invariant. For this reason, the local phase transformation is also called the gauge transformation. The modified Lagrangian reads

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^\mu A_\mu\psi = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4.48}$$

with the original partial differentiation operator ∂_μ replaced by a covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu \tag{4.49}$$

and the gauge field transforms as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{4.50}$$

e represents the gauge coupling.

Another gauge-invariant quantity, known as the field strength, can be constructed purely out of gauge fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.51}$$

We further construct a Lorentz scalar out of the field strength, so that we can include this in the Lagrangian, and this is the electromagnetism Lagrangian

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.52}$$

the factor of $-1/4$ is chosen such that it yields the correct normalization for the equation of motion of an applied current $\partial_\mu F^{\mu\nu} = j^\nu$.

The whole Lagrangian, describing a system of electrons and gauge photons interacting with each other, reads

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu A_\mu\psi \tag{4.53}$$

The first two terms are nothing but the same as a free theory of electrons and photons without interactions, while the last term shows that the electrons and photons are coupled through an interaction with a dimensionless coupling parameter e. If the parameter e is small, we can study the interacting theory by perturbative techniques in terms of Feynman diagrams.

4.2.2 Partition function for photons

Before going on, we briefly review the necessity of introducing gauge fixing and ghost terms into the formulation of the functional integral representation for photons. We illustrate this by computing the photon partition function. Let us start with the vacuum case,

$$Z = \int \mathcal{D}A_\mu \exp\left\{ i \int_0^{-i\beta} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right\} \tag{4.54}$$

Gauge transformations should not change anything physically, but we need to fix a gauge. The covariant gauge condition

$$G(A) = \partial_\mu A^\mu = w(x) \tag{4.55}$$

can be imposed by inserting the identity

$$1 = \int \mathcal{D}\alpha(x) \delta\big(G(A,\alpha) - w(x)\big) \left| \frac{\delta\big(G(A,\alpha) - w(x)\big)}{\delta\alpha} \right| \tag{4.56}$$

$w(x)$ is some funtion of x we can choose for the convenience of calculations. Take $w = 0$ we recover the Lorenz gauge condition.

There is still a residual gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{4.57}$$

which leaves the field strength unchanged, and thus it does not change the Lagrangian. We can write

$$G(A,\alpha) = \partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha \tag{4.58}$$

The determinant term in (4.56) is

$$\frac{\delta G\big(A, \alpha(x)\big)}{\delta\alpha(y)} = \partial_\mu \partial^\mu \delta^{(4)}(x-y) \tag{4.59}$$

Therefore, we write the partition function

$$Z = \int \mathcal{D}A\mathcal{D}\alpha \delta\big(G(A,\alpha) - w(x)\big) \left| \frac{\delta\big(G - w(x)\big)}{\delta\alpha} \right| \exp\left\{ i \int_0^{-i\beta} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right\} \tag{4.60}$$

Now we shift $A_\mu \to A_\mu + \partial_\mu \alpha$ and this shall not change the partition function

$$Z = \int \mathcal{D}A\mathcal{D}\alpha \delta\big(G(A) - w(x)\big) \det(\partial^2) \exp\left\{ i \int_0^{-i\beta} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right\} \tag{4.61}$$

but the integrand does not contain α, so we can take out the integration over $\int \mathcal{D}\alpha$. This integral is divergent, and it merely shows the redundancy of the residual gauge transformation.

We further average over $w(x)$ around zero with a Gaussian width ξ

$$\int \mathcal{D}w \frac{1}{\sqrt{2\pi\xi}} \mathrm{e}^{w^2(x)/2\xi} \tag{4.62}$$

where ξ is a gauge fixing parameter that one can choose for the convenience of calculations. The factor of $1/\sqrt{2\pi\xi}$ is a choice for the sake of normalization. We shall ignore it for a while until when it is necessary to put it back. Integration over $\mathcal{D}w$ can be performed with the delta function that we introduced to impose gauge conditions. Then the partition function embraces a gauge fixing term

$$\int \mathcal{D}A \exp\left\{ i \int_0^{-i\beta} d^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{(\partial_\mu A^\mu)^2}{2\xi} \right] \right\} \tag{4.63}$$

There is yet a $\det \partial^2 \delta^{(4)}(x-y)$ term. Using the representation for delta function

$$\delta^{(4)}(x-y) = \frac{1}{\beta V} \sum_{n,\mathbf{p}} \mathrm{e}^{i\omega_n(\tau_x - \tau_y) + i\mathbf{p}(\mathbf{x}-\mathbf{y})} \tag{4.64}$$

and acting the derivative operation on this, we get

$$\partial^2 \delta^{(4)}(x-y) = \frac{1}{\beta V} \sum_{n,\mathbf{p}} (-\omega_n^2 - \mathbf{p}^2) \exp\{ i\omega_n(\tau_x - \tau_y) + i\mathbf{p}(\mathbf{x}-\mathbf{y}) \} \tag{4.65}$$

We see that the $\det \partial^2$ term is not diagonal in the $x-y$ space, and moreover it is temperature dependent when expressed in frequency-momentum space. Therefore in the computation of the partition function, it cannot be simply ignored. We shall see how this term plays a crucial role to cancel the redundant degrees of freedom.

The $\det \partial^2$ term can be written in terms of a functional integral over Grassmannian variables by noting (4.25). We have

$$\det \partial^2 = \int \mathcal{D}\eta^* \mathcal{D}\eta \exp\left(-\int_0^{-i\beta} \eta^* \partial^2 \eta \right) \tag{4.66}$$

η^* and η are called the ghost fields.

Hence, the full partition function takes the form

$$Z = \int \mathcal{D}A \mathcal{D}\eta^* \mathcal{D}\eta \exp\left\{ i \int_0^{-i\beta} d^4x \big(\mathcal{L}_\gamma + \mathcal{L}_{gf} + \mathcal{L}_{gh} \big) \right\} \tag{4.67}$$

where the gauge fixing term and the ghost term are

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2 \tag{4.68}$$

$$\mathcal{L}_{gh} = -\eta^* \partial^2 \eta \tag{4.69}$$

Let us evaluate (4.67) more closely. Pick out the original Lagrangian $\mathcal{L}_\gamma$ and the gauge fixing term $\mathcal{L}_{gf}$, integrate by parts and drop the terms that vanish at infinity, one derives

$$\int \mathcal{D}A \exp\left\{\frac{i}{2}\int_0^{-i\beta} d^4x A^\mu \left[g_{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] A^\nu\right\} \tag{4.70}$$

We want to rotate to imaginary time variable $\tau = it = ix^0$ to compute in the finite temperature case. We have

$$\int \mathcal{D}A \exp\left\{-\frac{1}{2}\int_0^{\beta} d\tau d^3\mathbf{x} A^\mu \left[g_{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] A^\nu\right\} \tag{4.71}$$

If we want to write the Lagrangian in terms of τ, then we will get imaginary terms in the Lagrangian from the derivative $\partial/\partial x^0$. To avoid this happening, we make a unitary transformation for the photon fields $A^4 = iA^0$ and denote the photon fields as A^i, where the index i runs from $1,2,3,4$. One finds some useful conversion rules are

$$\delta^{ij} \leftrightarrow -g^{\mu\nu}, \quad A^iA^j \leftrightarrow A^\mu A^\nu \qquad \text{with } i,j = 1,2,3,4, \text{ and } \mu,\nu = 0,1,2,3 \tag{4.72}$$

With these substitutions into (4.71), it becomes

$$\int \mathcal{D}A \exp\left\{-\frac{1}{2}\int_0^{\beta} d\tau d^3\mathbf{x} A^i \left[\delta_{ij}\Box_\tau - \left(1-\frac{1}{\xi}\right)\partial_i\partial_j\right] A^j\right\} \tag{4.73}$$

The gauge particles are bosons and they naturally have the Fourier expansion

$$A^i(\tau,\mathbf{x}) = \sqrt{\frac{\beta}{V}}\sum_{n,\mathbf{k}} \mathrm{e}^{i(\omega_n\tau + \mathbf{k}\cdot\mathbf{x})} A^i_{n,\mathbf{k}} \tag{4.74}$$

Going to the frequency-momentum space, we find the partition function becomes

$$\begin{aligned}
&\int \mathcal{D}A_{n,\mathbf{k}} \exp\left\{\sum_{n,\mathbf{k}} -\frac{\beta^2}{2} A^{i*}_{n,\mathbf{k}}\left[\delta_{ij}(\omega_n^2+\mathbf{k}^2) - \left(1-\frac{1}{\xi}\right)k_ik_j\right] A^j_{n,\mathbf{k}}\right\} \\
=&\int \mathcal{D}A_{n,\mathbf{k}} \exp\left\{\sum_{n,\mathbf{k}} -\frac{\beta^2}{2} A^{i*}_{n,\mathbf{k}} M_{ij}(\xi) A^j_{n,\mathbf{k}}\right\} \\
=&\prod_{n,\mathbf{k}}\sqrt{\frac{\pi^N}{\det M_\xi}}
\end{aligned} \tag{4.75}$$

where M is a 4×4 matrix of arguments ω_n and $\mathbf{k}$. For convenience we set $\mathbf{k} = (0,0,k)$, then this matrix can be explicitly written down as

$$M_\xi = \beta^2 \begin{pmatrix} \mathbf{k}^2 + \frac{\omega_n^2}{\xi} & 0 & 0 & -\left(1-\frac{1}{\xi}\right)\omega_n\mathbf{k} \\ 0 & \omega_n^2+\mathbf{k}^2 & 0 & 0 \\ 0 & 0 & \omega_n^2+\mathbf{k}^2 & 0 \\ -\left(1-\frac{1}{\xi}\right)\omega_n\mathbf{k} & 0 & 0 & \omega_n^2 + \frac{\mathbf{k}^2}{\xi} \end{pmatrix} \tag{4.76}$$

whose determinant is

$$\det M_\xi = \left\{ \left(\mathbf{k}^2 + \frac{\omega_n^2}{\xi}\right) \left(\omega_n^2 + \mathbf{k}^2\right)^2 \left(\omega_n^2 + \frac{\mathbf{k}^2}{\xi}\right) - \left[- \left(1 - \frac{1}{\xi}\right) \omega_n \mathbf{k} \right]^2 \left(\omega_n^2 + \mathbf{k}^2\right)^2 \right\}$$

$$= \frac{\left[\beta^2 \left(\omega_n^2 + \mathbf{k}^2\right)\right]^4}{\xi} \tag{4.77}$$

from which we obtain the contribution to the partition function from $\mathcal{L}_\gamma$ and $\mathcal{L}_{gf}$

$$\ln Z_{(\gamma+gf)} = -\frac{1}{2} \sum_{n,\mathbf{k}} \ln \left[\beta^2 \left(\omega_n^2 + \mathbf{k}^2\right)\right] \times 4 \tag{4.78}$$

The ξ-dependence is canceled by the factor of $(1/\sqrt{2\pi\xi})^N$ originating from the introduction of Gaussian integral (4.62) at each location of x. Since the ghost piece is independent of gauge fixing parameter ξ, so the full partition function, and therefore all the thermodynamicals must be gauge fixing independent.

We shall also take the ghost fields into account. Its contribution to the partition function is

$$\int \mathcal{D}\eta^* \mathcal{D}\eta \exp \left\{ -i \int_0^{-i\beta} d^4x \eta^* \partial^2 \eta \right\} = \int \mathcal{D}\eta^* \mathcal{D}\eta \exp \left\{ \int_0^\beta d\tau d^3\mathbf{x} \eta^* \Box_\tau \eta \right\} \tag{4.79}$$

Substituting the Fourier expansion of the ghost fields

$$\eta(\mathbf{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n,\mathbf{k}} \mathrm{e}^{i(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})} \eta_{n,\mathbf{k}} \tag{4.80}$$

We obtain the following expression

$$\int \mathcal{D}\eta^*_{-n,-\mathbf{k}} \mathcal{D}\eta_{n,\mathbf{k}} \exp \left\{ -\sum_{n,\mathbf{k}} \beta^2 \eta^*_{-n,-\mathbf{k}} (\omega_n^2 + \mathbf{k}^2) \eta_{n,\mathbf{k}} \right\} = \prod_{n,\mathbf{k}} \beta^2 (\omega_n^2 + \mathbf{k}^2) \tag{4.81}$$

Hence the ghost fields contribute to the partition function

$$\ln Z_{gh} = -\frac{1}{2} \sum_{n,\mathbf{k}} \ln \left[\beta^2 \left(\omega_n^2 + \mathbf{k}^2\right)\right] \times (-2) \tag{4.82}$$

Putting all the contributions to the photon partition function together, we get

$$\ln Z = -\frac{1}{2} \sum_{n,\mathbf{k}} \ln \left[\beta^2 \left(\omega_n^2 + \mathbf{k}^2\right)\right] \times 2 \tag{4.83}$$

We should note that in this covariant gauge, we would render 4 degrees of freedom in comparison with (3.17) if the ghost field contributions are not included. There would be two physical transverse degrees of freedom, but also two unwanted degrees of freedom relating to the unphysical time-like photons and longitudinal photons. The two unwanted degrees of freedom are exactly canceled by the ghost contributions.

4.3 Photon propagator

4.3.1 Vacuum photon propagator

We are not going to study the photon propagators. We define the photon propagator in imaginary time

$$D^{\mu\nu}(\mathbf{x},\tau) = \frac{\int \mathcal{D}A A^{\mu}(\mathbf{x},\tau)A^{\nu}(0,0)\exp\left\{\int_0^\beta d\tau d^3\mathbf{x}\mathcal{L}\right\}}{\int \mathcal{D}A \exp\left\{\int_0^\beta d\tau d^3\mathbf{x}\mathcal{L}\right\}} \tag{4.84}$$

Recall a useful technique that the n-point functions at zero temperature can be derived from the generating functional

$$Z[j] = \int \mathcal{D}A \exp\left\{i\int d^4x(\mathcal{L} + A^{\mu}j_{\mu})\right\} \tag{4.85}$$

The n-point correlators are given by the derivatives of the generating functional with respect to j^{μ}'s, for example,

$$\langle A^{\mu}\rangle = \frac{1}{Z[j]}\left(-i\frac{\delta}{\delta j_{\mu}}\right) Z[j]\Big|_{j=0} \tag{4.86}$$

$$\langle A^{\mu}A^{\nu}\rangle = \frac{1}{Z[j]}\left(-i\frac{\delta}{\delta j_{\mu}}\right)\left(-i\frac{\delta}{\delta j_{\nu}}\right) Z[j]\Big|_{j=0} \tag{4.87}$$

Converting this method to our discussions for finite temperature field theories, the corresponding generating functional is

$$Z[j] = \int \mathcal{D}A \exp\left\{\int_0^\beta d\tau d^3\mathbf{x}(\mathcal{L} + A^{\mu}j_{\mu})\right\} \tag{4.88}$$

The factor of $(-i)$ is absorbed into the imaginary time variable, so there is no need to assign this factor to the derivative operators to generate the n-point functions.

For the pure electromagnetic field, the argument of the exponential in the generating functional is

$$\frac{1}{2}A_i\left[\Box_{\tau}\delta_{ij} - \left(1-\frac{1}{\xi}\right)\partial_i\partial_j\right]A_j + A_i j_i = \frac{1}{2}A_i G_{ij} A_j + A_i j_i \tag{4.89}$$

The path integral over $\mathcal{D}A$ is over the full space, so we can shift the A fields

$$A_i(x) \to A_i(x) - \int_0^\beta d\tau' d^3\mathbf{y}\left[G_{ij}^{-1}(x-y)\right]j_j(y) \tag{4.90}$$

We find the exponential term (4.89) is translated into

$$\int_0^\beta d\tau d^3\mathbf{x}\frac{1}{2}\left\{A_i(x)G_{ij}(x)A_j(x) - \int_0^\beta d\tau' d^3\mathbf{y} j_k(x)G_{kl}^{-1}(x-y)j_l(y)\right\} \tag{4.91}$$

Therefore, the two-point function reads

$$\langle A_i(\mathbf{x},\tau_x)A_j(\mathbf{y},\tau_y) = \frac{\delta}{\delta j_x^i}\frac{\delta}{\delta j_y^j}\left\{-\frac{1}{2}\int_0^\beta d\tau_x d^3\mathbf{x}\int_0^\beta d\tau_y d^3\mathbf{y} j^k(x)(G^{-1})^{kl}(x-y)j^l(y)\right\}\bigg|_{j=0}$$
$$= -\frac{1}{2}\left[(G^{-1})^{ij}(x-y)+(G^{-1})^{ij}(y-x)\right] \tag{4.92}$$

Let's evaluate $(G^{-1})^{ij}(x-y)$. In frequency-momentum space, we have

$$G^{ij}(k) = -\delta^{ij}(\omega_n^2+\mathbf{k}^2)+\left(1-\frac{1}{\xi}\right)k^ik^j \tag{4.93}$$

The inverse of this is solved to be

$$(G^{-1})^{jk}(k) = -\frac{\delta^{jk}}{\omega_n^2+\mathbf{k}^2}+\frac{(1-\xi)k^jk^k}{(\omega_n^2+\mathbf{k}^2)^2} \tag{4.94}$$

One may check this is indeed the inverse of $G^{ij}(k)$, satisfying

$$G^{ij}(G^{-1})^{jk} = \delta^{ik} \tag{4.95}$$

The free photon propagator in Euclidean space has the Fourier transform

$$\langle A^i(x)A^j(y)\rangle_\beta = \frac{1}{\beta V}\sum_{n,\mathbf{k}} \mathrm{e}^{i\omega_n(\tau_x-\tau_y)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}D^{ij}(\omega_n,\mathbf{k}) \tag{4.96}$$

So the Euclidean photon propagator in energy-momentum space

$$D^{ij}(\omega_n,\mathbf{k}) = -(G^{-1})^{ij}(x-y) = \frac{\delta^{ij}}{\omega_n^2+\mathbf{k}^2}-(1-\xi)\frac{k^ik^j}{(\omega_n^2+\mathbf{k}^2)^2} \tag{4.97}$$

Remember that the Euclidean indices i and j run from 1, 2, 3, 4, and the definitions $A_4 = iA_0$. We can define $k_4 = ik_0 = \omega_n$, so that k^i has the same conversion relation as A^i in (4.72). Therefore, the photon propagator in Minkowski space-time is

$$D^{\mu\nu}(\omega_n,\mathbf{k}) = \frac{-g^{\mu\nu}}{\omega_n^2+\mathbf{k}^2}-(1-\xi)\frac{k^\mu k^\nu}{(\omega_n^2+\mathbf{k}^2)^2} \tag{4.98}$$

The analytic continuation to real k^0 can be worked out

$$\begin{aligned}
\langle A^\mu(x)A^\nu(y)\rangle_\beta &= \frac{1}{\beta V}\sum_{n,\mathbf{k}} D^{\mu\nu}(\omega_n,\mathbf{k})\mathrm{e}^{i\omega_n(\tau_x-\tau_y)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{1}{\beta V}\sum_{n,\mathbf{k}}\left\{\frac{-g^{\mu\nu}}{-(k^0)^2+\mathbf{k}^2}-(1-\xi)\frac{k^\mu k^\nu}{[-(k^0)^2+\mathbf{k}^2]^2}\right\} \\
&\quad \times \mathrm{e}^{-ik^0(x^0-y^0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{1}{\beta V}\sum_{n,\mathbf{k}}\left\{\frac{g^{\mu\nu}}{k^2}-(1-\xi)\frac{k^\mu k^\nu}{(k^2)^2}\right\}\mathrm{e}^{ik(x-y)}
\end{aligned} \tag{4.99}$$

The free photon propagator can be written as

$$D_F^{\mu\nu}(k^0,\mathbf{k}) = \frac{i}{k^2}\left[-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \tag{4.100}$$

4.3.2 Longitudinal and transverse projections

By far we have only discussed the free photon propagator in vacuum. But in the QED theory, photons are interactive with electrons. The behavior of the propagation of photons will be modified in the existence of a medium. The full photon propagator will take the form

$$D^{\mu\nu} = D_F^{\mu\nu} + D_F^{\mu\alpha}(-i\Pi_{\alpha\beta})D_F^{\beta\nu} + D_F^{\mu\alpha}(-i\Pi_{\alpha\beta})D_F^{\beta\gamma}(-i\Pi_{\gamma\delta})D_F^{\delta\nu} + \cdots \tag{4.101}$$

where $\Pi_{\mu\nu}$ is called the *photon self energy.* This expression could be simplified by noting that in vacuum $\Pi^{\mu\nu}$ obeys the Ward identity

$$k_\mu \Pi^{\mu\nu} = 0 \tag{4.102}$$

$\Pi^{\mu\nu}$ is Lorentz covariant, so

$$\Pi^{\mu\nu} = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right)\Pi(k) \tag{4.103}$$

The presence of a medium will not affect the Ward identity, but will break the Lorentz covariance. Another way of saying this is that photons do not have a preferred rest frame in vacuum, but the medium introduces another 4-vector $n^\mu = (1, 0, 0, 0)$ into the problem. We write

$$\Pi^{\mu\nu} = ag^{\mu\nu} + bk^\mu k^\nu + cn^\mu k^\nu + dk^\mu n^\nu + en^\mu n^\nu \tag{4.104}$$

where a, b, c, d, e are some numerical factors that can be determined. Note that $n \cdot k = k^0 \neq 0$, thus not all the components are independent.

Consider the effect of k/mu acting on the tensor $n_\mu n_\nu$,

$$\begin{aligned} k^\mu(n_\mu n_\nu) &= k^0 n_\nu \\ k^\mu(k_\mu n_\nu) &= k^2 n_\nu \\ k^\mu(n_\mu k_\nu) &= k^0 k_\nu \end{aligned} \tag{4.105}$$

We can construct tensors out of these combinations. A particular combination of our interest is

$$\begin{aligned} P_T^{\mu\nu} &= -g^{\mu\nu} + n^\mu n^\nu + b(k^\mu - k^0 n^\mu)(k^\nu - k^0 n^\nu) \\ &= \delta^{ij} g_i^\mu g_j^\nu + bk^i k^j \delta_i^\mu \delta_j^\nu \end{aligned} \tag{4.106}$$

where b is some undetermined constant. If we set

$$b = -\frac{1}{\mathbf{k}^2} \tag{4.107}$$

then we obtain

$$P_T^{\mu\nu} = \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)\delta_i^\mu \delta_j^\nu \tag{4.108}$$

It can be immediately verified that

$$k_\mu P_T^{\mu\nu} = 0 \tag{4.109}$$

which means that $P_T^{\mu\nu}$ is orthogonal to the 4-vector k_μ as well as the 3-vector $\mathbf{k}$. Therefore it is called the *transverse projector* (transverse to $\mathbf{k}$).

The other projector, the *longitudinal projector*, is defined as

$$P_L^{\mu\nu} = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mathbf{k}^2}\right) - P_T^{\mu\nu} \tag{4.110}$$

It is three-dimensionally longitudinal, but still four-dimensionally transverse.

These two projectors have the following properties,

$$g_{\mu\nu} P_T^{\mu\nu} = -3 + 1 = -2 \tag{4.111}$$

$$g_{\mu\nu} P_L^{\mu\nu} = -4 + 1 + 2 = -1 \tag{4.112}$$

corresponding to 2 transverse projectors and 1 longitudinal projector, respectively.

We can also check that

$$P_T^{\mu\alpha} P_{T\alpha\nu} = P^\mu_{T\nu} \tag{4.113}$$

$$P_L^{\mu\alpha} P_{L\alpha\nu} = P^\mu_{L\nu} \tag{4.114}$$

$$P_T^{\mu\alpha} P_{L\alpha\nu} = 0 \tag{4.115}$$

showing that these are indeed projectors.

4.3.3 Full photon propagator

Using these properties of transverse and longitudinal projectors, we can decompose the free propagator as

$$D_F^{\mu\nu}(k) = i\frac{-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2} \equiv i\frac{P_T^{\mu\nu}}{k^2} + i\frac{P_L^{\mu\nu}}{k^2} - i\xi\frac{k^\mu k^\nu}{k^2} \tag{4.116}$$

We also decompose the photon self energy

$$\Pi^{\mu\nu} = F P_L^{\mu\nu} + G P_T^{\mu\nu} \tag{4.117}$$

F and G are some scalar functions to be determined. We then have

$$D_F^{\mu\alpha}\Pi_{\alpha\beta}D_F^{\beta\nu} = F\frac{iP_L^{\mu\nu}}{k^2} + G\frac{iP_T^{\mu\nu}}{k^2} \tag{4.118}$$

This is summed to get the full propagator of the form

$$D^{\mu\nu} = i\frac{P_L^{\mu\nu}}{k^2 - F} + i\frac{P_T^{\mu\nu}}{k^2 - G} \tag{4.119}$$

If we have expressions for the decomposition functions F and G, then we can solve the poles of these propagators and understand the kind of modes they have.

4.4 Photon self energy

From now on in this and the next section, we are going to use a different notation to distinguish 4-momenta and 3-momenta. Any 4-momentum will be denoted by capital letters, and the 3-momentum will be denoted by the corresponding lower case letters. For example,

$$K^\mu = (k^0, \mathbf{k}) \tag{4.120}$$

The modulus of a 3-momentum $\mathbf{k}$ is simply represented by k.

4.4.1 Photon self energy

In order to learn about the behavior of the full propagator, we want to compute at finite temperature the photon self energy $\Pi^{\mu\nu}(k^0, \mathbf{k})$, where the Matsubara frequencies $k^0 = 2n\pi/\beta$. It can be shown that the evaluation of $\Pi^{\mu\nu}$ is closely related to that of functions F and G.

Let us first study the photon self energy at the lowest order. The relevant one-loop Feynman diagram is shown in Fig.1.

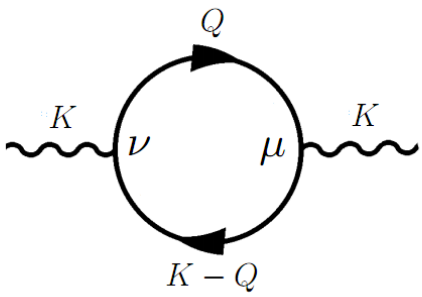


Figure 1: One-loop photon self energy

Using the Feynman rules, this corresponds to

$$\Pi^{\mu\nu}(k^0 = i\omega_n, \mathbf{k}) = -\frac{e^2}{\beta}\sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathrm{Tr}\left\{\gamma^\mu S(Q)\gamma^\nu S(K-Q)\right\} \tag{4.121}$$

There should be a minus sign corresponding to the closed fermion loop.

Note that fermion propagator can be decomposed, for example

$$S(Q) = (\not{Q} - m)\tilde{\Delta}(i\omega_n, \mathbf{q}) \tag{4.122}$$

Similar relation hold for $S(K-Q)$. So we can write

$$\Pi^{\mu\nu} = e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathrm{Tr}\{\gamma^\mu(\not{Q}-m)_{s_1}\gamma^\nu(\not{K}-\not{Q}-m)_{s_2}\}\frac{1}{\beta}\sum_n \tilde{\Delta}(i\omega_n, \mathbf{q})\tilde{\Delta}(i(\omega_m-\omega_n), \mathbf{k}-\mathbf{q}) \tag{4.123}$$

The frequency sum can be performed by using the relation (A.27). Therefore we get a expression for the photon self energy

$$\Pi^{\mu\nu} = e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\mathrm{Tr}\{\gamma^\mu(\not{Q} - m)_{s_1}\gamma^\nu(\not{K} - \not{Q} - m)_{s_2}\}}{4E_q E_{k-q}} \frac{1 - \tilde{n}(-s_1 E_q) - \tilde{n}(-s_2 E_{k-q})}{i\omega_n - s_1 E_q - s_2 E_{k-q}} \tag{4.124}$$

The summation over $s_1, s_2 = \pm$ is omitted.

4.4.2 Computation of F

We want to compute F and G to obtain a physical interpretation of the photon self energy. With the notation (4.120), we note the projectors

$$P_T^{\mu\nu} = \left(\delta^{ij} - \frac{K^i K^j}{k^2}\right) g_i^\mu g_j^\nu \tag{4.125}$$

$$P_L^{\mu\nu} = \left(-g^{\mu\nu} + \frac{K^\mu K^\nu}{(k^0)^2 - k^2}\right) - P_T^{\mu\nu} \tag{4.126}$$

whose 00-components read

$$P_T^{00} = 0, \qquad P_L^{00} = -1 + \frac{k^0 k^0}{(k^0)^2 - k^2} = \frac{k^2}{(k^0)^2 - k^2} \tag{4.127}$$

Recall that $\Pi^{\mu\nu} = F P_L^{\mu\nu} + G P_T^{\mu\nu}$, so we have

$$F = \frac{(k^0)^2 - k^2}{k^2} \Pi^{00} \tag{4.128}$$

relating the calculation of F closely to the evaluation of Π^{00}.

We have already had an expression (4.124) for $\Pi^{\mu\nu}$. However, the complete evaluation is rather complicated, and we shall step back and limit to the high-temperature limit, also known as the hard thermal loop (HTL) limit. The significance of this particular limit will be discussed in the last section. By saying the HTL limit, we actually mean that the temperature is much higher than any mass scale at zero temperature. All these mass terms can be dropped, and external momenta K can be neglected with comparison to the loop momenta Q.

Within the HTL limit, dropping the mass terms we have

$$\Pi^{00} = -e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\mathrm{Tr}\{\gamma^0 \not{Q}_{s_1} \gamma^0 (\not{K} - \not{Q})_{s_2}\}}{4E_q E_{k-q}} \frac{1 - \tilde{n}(-s_1 E_q) - \tilde{n}(-s_2 E_{k-q})}{k^0 - s_1 E_q - s_2 E_{k-q}} \tag{4.129}$$

Using the γ-matrices identities, we get

$$\mathrm{Tr}\{\gamma^0 \not{Q}_{s_1} \gamma^0 (\not{K} - \not{Q})_{s_2}\} = 4Q_{s_1}^\mu (K-Q)_{s_2}^\nu + 4Q_{s_1}^\nu (K-Q)_{s_2}^\mu - 4g^{\mu\nu} Q_{s_1}(K-Q)_{s_2} \tag{4.130}$$

Also under the HTL approximation, we have

$$
\begin{aligned}
Q_{s_1} &= (E_q, s_1\mathbf{q}) \approx (q, s_1\mathbf{q}) \\
(K-Q)_{s_2} &= (E_{\mathbf{k-q}}, s_2(\mathbf{k}-\mathbf{q})) \approx (q, -s_2\mathbf{q})
\end{aligned} \tag{4.131}
$$

The numerator of in the expression for Π^{00} becomes

$$
4\left[q^2 + q^2 - (1+s_1 s_2)q^2\right] = 4q^2(1 - s_1 s_2) \tag{4.132}
$$

So the only non-zero terms are $s_1 = -s_2 = \pm$.

$$
\begin{aligned}
\Pi^{00} &= -e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{4q^2(1+1)}{4q^2} \frac{1 - \tilde{n}(-sE_q) - \tilde{n}(sE_{k-q})}{k^0 - sE_q + sE_{k-q}} \\
&= -2e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left\{ \frac{1 - \tilde{n}(-E_q) - \tilde{n}(E_{k-q})}{k^0 - E_q + E_{k-q}} + \frac{1 - \tilde{n}(E_q) - \tilde{n}(-E_{k-q})}{k^0 + E_q - E_{k-q}} \right\}
\end{aligned} \tag{4.133}
$$

In the HTL limit, we have $E_q \approx q$, and $E_{k-q} \approx q - k\cos\theta$. Remember that here we simply denoted $|\mathbf{q}|$ as q for the convenience of writing, which shall not cause confusion with our notation (4.120).

Ignore all the thermal independent factors, and pick out only the thermal contributions, we obtain

$$
\begin{aligned}
\Pi^{00} &= -2e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left\{ \frac{\tilde{n}(q) - \tilde{n}(q - k\cos\theta)}{k^0 - q + q - k\cos\theta} + \frac{\tilde{n}(q - k\cos\theta) - \tilde{n}(q)}{k^0 + q - q + k\cos\theta} \right\} \\
&= -2e^2 \int \frac{dq q^2 d\cos\theta}{(2\pi)^2} \left\{ \frac{\frac{\partial\tilde{n}}{\partial q} k\cos\theta}{k^0 - k\cos\theta} - \frac{\frac{\partial\tilde{n}}{\partial q} k\cos\theta}{k^0 + k\cos\theta} \right\}
\end{aligned} \tag{4.134}
$$

where in the last step, the integral $\int d\phi = 2\pi$ in $\int d^3\mathbf{q}$ was taken, and the assumption was used that the internal momentum q of the loop is much larger than the external momentum k.

Integration over dq can be done by evaluating

$$
\begin{aligned}
\int_0^\infty dq q^2 \frac{\partial\tilde{n}(q)}{\partial q} &= q^2\tilde{n}(q)\big|_0^\infty - 2\int_0^\infty dq q\tilde{n}(q) \\
&= 0 - 2 \times \frac{1}{12}\pi^2 T^2 = -\frac{1}{6}\pi^2 T^2
\end{aligned} \tag{4.135}
$$

Substituting and we have

$$
\begin{aligned}
\Pi^{00} &= e^2 \frac{\pi^2 T^2}{6} \frac{1}{4\pi^2} \int d\cos\theta \left\{ \frac{k\cos\theta}{k^0 - k\cos\theta} - \frac{k\cos\theta}{k^0 + k\cos\theta} \right\} \\
&= \frac{e^2 T^2}{12} \int d\cos\theta \left\{ -2 + \frac{k^0}{k^0 - k\cos\theta} - \frac{k^0}{k^0 + k\cos\theta} \right\}
\end{aligned} \tag{4.136}
$$

There is only one integration over $d\theta$ left. We can evaluate the integral by defining a new parameter $y = \cos\theta$, and finally we compute

$$\begin{aligned}
&= \frac{e^2T^2}{12} \times (-4) + \frac{e^2T^2}{12} \int_{-1}^{1} dy \left\{ \frac{k^0}{k^0 - ky} - \frac{k^0}{k^0 + ky} \right\} \\
&= -\frac{e^2T^2}{3} + \frac{e^2T^2}{6} \int_{-1}^{1} dy \frac{k^0}{k^0 - ky} \\
&= -\frac{e^2T^2}{3} + \frac{e^2T^2}{6} \ln\left(\frac{k^0 + k}{k^0 - k}\right) \frac{k^0}{k} \\
&= -2m^2 + m^2 \ln\left(\frac{k^0 + k}{k^0 - k}\right) \frac{k^0}{k}
\end{aligned} \tag{4.137}$$

The term $m^2 = e^2T^2/6$ is identified as the *photon thermal mass* squared.

Instead of using the logarithm function, sometimes we also use the Legendre function of the second kind.

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) \tag{4.138}$$

So we more compactly write

$$\Pi^{00} = -2m^2 \left[1 - \frac{k^0}{k} Q_0\left(\frac{k^0}{k}\right)\right] \tag{4.139}$$

The decomposition function F can be immediately obtained by plugging the expression for Π^{00} into (4.128)

$$F = -2m^2 \frac{(k^0)^2 - k^2}{k^2} \left[1 - \frac{k^0}{k} Q_0\left(\frac{k^0}{k}\right)\right] \tag{4.140}$$

4.4.3 Computation of G

We can also compute the scalar function G. It can be shown that the computation of G is closely related to the evaluation of Π^{xx}. If we take the xx-component of the projectors $P^{\mu\nu}_{L,T}$

$$P^{xx}_T = 1, \qquad P^{xx}_L = 0 \tag{4.141}$$

from which we immediately yield

$$G = \Pi^{xx} \tag{4.142}$$

So next we will take some efforts to calculate Π^{xx}.

We limit ourselves to the HTL approximations as before. Going back to (4.124), from the trace, we get

$$4(s_1 q^x)(-s_2 q^x) + 4(-s_2 q^x)(s_1 q^x) + 4q^2(1 + s_1 s_2) \tag{4.143}$$

with $q^x = q\sin\theta\cos\phi$, this is

$$4q^2\left[1 + s_1 s_2(1 - 2\sin^2\theta\cos^2\phi)\right] \tag{4.144}$$

The integration over $d\phi$ can be carried out, and we are left with

$$4q^2\left[1 + s_1 s_2(1 - 2\sin^2\theta)\right]2\pi = 4q^2(1 + s_1 s_2\cos^2\theta)2\pi \tag{4.145}$$

The terms are of two types

$$s_1 = s_2 \to 4q^2(1 + \cos^2\theta)2\pi \tag{4.146}$$

$$s_1 = -s_2 \to 4q^2(1 - \cos^2\theta)2\pi \tag{4.147}$$

From the second fraction in Π^{xx}, there are also two possible cases

$$s_1 = s_2 \to d_1 = \frac{1 - \tilde{n}(-E_q) - \tilde{n}(-E_{k-q})}{k^0 - E_q - E_{k-q}} + \frac{1 - \tilde{n}(E_q) - \tilde{n}(E_{k-q})}{k^0 + E_q + E_{k-q}} \tag{4.148}$$

$$s_1 = -s_2 \to d_2 = \frac{1 - \tilde{n}(-E_q) - \tilde{n}(E_{k-q})}{k^0 - E_q + E_{k-q}} + \frac{1 - \tilde{n}(E_q) - \tilde{n}(-E_{k-q})}{k^0 + E_q - E_{k-q}} \tag{4.149}$$

In the case of HTL limit and noting that

$$\tilde{n}(-E) = \frac{1}{\mathrm{e}^{-\beta E} + 1} = \frac{\mathrm{e}^{\beta E}}{\mathrm{e}^{\beta E} + 1} = 1 - \tilde{n}(E) \tag{4.150}$$

we see the $s_1 = \pm s_2$ pieces have different behaviors

$$\begin{aligned} d_1 &= \frac{\tilde{n}(q) + \tilde{n}(q - k\cos\theta) - 1}{k^0 - q - q + k\cos\theta} + \frac{1 - \tilde{n}(q) - \tilde{n}(q - k\cos\theta)}{k^0 + q + q - k\cos\theta} \\ &\sim \frac{2\tilde{n}(q)}{-2q} - \frac{2\tilde{n}(q)}{2q} = -\frac{2\tilde{n}(q)}{q} \end{aligned} \tag{4.151}$$

where the $\sim$ means that we keep only the thermal contributions of interest. Note that in the numerators of d_1, the statistical distribution factors are of the same sign. Also we have

$$\begin{aligned} d_2 &= \frac{\tilde{n}(q) - \tilde{n}(q - k\cos\theta)}{k^0 - q + q - k\cos\theta} + \frac{\tilde{n}(q - k\cos\theta) - \tilde{n}(q)}{k^0 + q - q + k\cos\theta} \\ &= \frac{\frac{\partial\tilde{n}}{\partial q}k\cos\theta}{k^0 - k\cos\theta} + \frac{\frac{-\partial\tilde{n}}{\partial q}k\cos\theta}{k^0 + k\cos\theta} \\ &= \frac{\partial\tilde{n}}{\partial q}\left[-1 + \frac{k^0}{k^0 - k\cos\theta} - 1 + \frac{k^0}{k^0 + k\cos\theta}\right] \\ &= \frac{\partial\tilde{n}}{\partial q}\left[-2 + 2\frac{k^0}{k^0 - k\cos\theta}\right] \end{aligned} \tag{4.152}$$

where the last step is only valid under the integration $\int d\cos\theta$, and the numerators of d_2 include the difference of statistical distribution factors.

Putting all the pieces together, we obtain

$$\begin{aligned}\Pi^{xx} &= -e^2 \int \frac{dqq^2 d\cos\theta}{(2\pi)^2 4q^2}\Big[\big(\text{terms from } s_1 = s_2\big) + \big(\text{terms from } s_1 = -s_2\big)\Big] \\ &= -e^2 \int_{-1}^{1} dy \int \frac{dqq^2}{(2\pi)^2}\left[(1+y^2)\frac{-2\tilde{n}(q)}{q} + (1-y^2)\frac{\partial \tilde{n}(q)}{\partial q}\left(-2 + \frac{2k^0}{k^0 - ky}\right)\right]\end{aligned} \tag{4.153}$$

We integrate over dq by using the following identities

$$\int dqq\tilde{n}(q) = \frac{\pi^2 T^2}{12} \qquad \int dqq^2 \frac{\partial \tilde{n}(q)}{\partial q} = -\frac{\pi^2 T^2}{6} \tag{4.154}$$

So we have

$$\begin{aligned}\Pi^{xx} &= e^2 \frac{\pi^2 T^2}{6}\frac{1}{4\pi^2}\int_{-1}^{1} dy \left[(1+y^2) + (1-y^2)\left(-2 + 2\frac{k^0}{k^0 - ky}\right)\right] \\ &= \frac{\pi^2 T^2}{6}\frac{1}{2}\int_{-1}^{1} dy \left(\frac{k^0}{k}\frac{1}{\frac{k^0}{k} - y} - \frac{k^0}{k}\frac{y^2}{\frac{k^0}{k} - y}\right)\end{aligned} \tag{4.155}$$

We have encountered the first integrand before, the result of this integral is

$$\frac{1}{2}\int_{-1}^{1} \frac{dy}{\frac{k^0}{k} - y} = Q_0\left(\frac{k^0}{k}\right) \tag{4.156}$$

while the second integrand can be manipulated

$$\begin{aligned}\int_{-1}^{1} dy \frac{y^2}{\frac{k^0}{k} - y} &= \int_{-1}^{1} dy \left[\left(\frac{k^0}{k} - y\right) - 2\frac{k^0}{k} + \left(\frac{k^0}{k}\right)^2 \frac{1}{\frac{k^0}{k} - y}\right] \\ &= -y\frac{k^0}{k}\Big|_{-1}^{1} + \left(\frac{k^0}{k}\right)^2 \int_{-1}^{1} \frac{dy}{\frac{k^0}{k} - y} \\ &= -2\frac{k^0}{k} + 2\left(\frac{k^0}{k}\right)^2 Q_0\left(\frac{k^0}{k}\right)\end{aligned} \tag{4.157}$$

With these results, we finally obtain

$$G = \Pi^{xx} = m^2\left(\frac{k^0}{k}\right)^2 + m^2\left(\frac{k^0}{k}\right)\left[1 - \left(\frac{k^0}{k}\right)^2\right] Q_0\left(\frac{k^0}{k}\right) \tag{4.158}$$

where the thermal mass squared is the same as before $m^2 = e^2 T^2/6$

We shall notice that F, G and thus the photon self energy are not only temperature dependent but also momentum dependent.

4.5 Photon Collective Modes

Now that we have the longitudinal and transverse decomposition functions F and G, we can express the in-medium soft photon propagator, or HTL propagator as,

$$D^{\mu\nu} = \frac{iP_L^{\mu\nu}}{(k^0)^2 - k^2 - F} + \frac{iP_T^{\mu\nu}}{(k^0)^2 - k^2 - G} \tag{4.159}$$

We can understand the kind of modes of this medium has by examining the solutions to the poles of the propagators. We may look at the the different limits of these modes at small or large values of momentum.

4.5.1 Longitudinal modes

For the longitudinal modes, the dispersion relation can be solved from

$$(k^0)^2 - k^2 + 2m^2 \frac{(k^0)^2 - k^2}{k^2} \left[1 - \frac{k^0}{k} Q_0 \left(\frac{k^0}{k} \right) \right] = 0 \tag{4.160}$$

or simply

$$1 + \frac{2m^2}{k^2} \left[1 - \frac{k^0}{k} Q_0 \left(\frac{k^0}{k} \right) \right] = 0 \tag{4.161}$$

For large momenta k, we may find to a first order approximation $\omega_L = k^0 \approx k \gg \omega_P$. Note that in this limit

$$Q_0 \left(\frac{k^0}{k} \right) \approx \frac{1}{2} \ln \frac{2k}{\omega_L - k} \gg 1 \tag{4.162}$$

so we only keep this term in the square brackets in (4.161) when solving it. Under these suitable approximations, we derive

$$1 - \frac{m^2}{k^2} \ln \frac{2k}{\omega_L - k} = 0 \tag{4.163}$$

The solution of which is easily found

$$\omega_L = k \left(1 + 2\mathrm{e}^{-k^2/m^2} \right) \tag{4.164}$$

For small value of k, i.e., for $k^0/k \gg 1$, we find the expansion for $Q_0(x)$

$$\begin{aligned} Q_0(x) &= \frac{1}{2} \ln \frac{x+1}{x-1} \\ &= \frac{1}{2} \left[\ln \left(1 + \frac{1}{x} \right) - \left(1 - \frac{1}{x} \right) \right] \\ &= \frac{1}{2} \left[\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + \cdots \right) - \left(-\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} - \cdots \right) \right] \\ &= \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots \end{aligned} \tag{4.165}$$

where the dots representing higher order terms. This expansion is valid for large x. Let us define the *plasma frequency* as

$$\omega_P^2 = \frac{2}{3} m^2 = \frac{1}{9} e^2 T^2 \tag{4.166}$$

We then rewrite the dispersion equation (4.161) for longitudinal excitations as

$$\omega_L^4 - \omega_P^2\omega_L^2 - \frac{3}{5}\omega_P^2 k^2 = 0 \tag{4.167}$$

which is solved to be

$$\begin{aligned}
\omega_L^2 &= \frac{1}{2}\left(\omega_P^2 + \sqrt{\omega^4 + 4\cdot\frac{3}{5}\omega_P^2 k^2}\right) \\
&= \frac{1}{2}\left[\omega_P^2 + \omega_P^2\left(1 + \frac{1}{2}\cdot 4\cdot\frac{3}{5}\omega_P^2 k^2\right)\right] \\
&= \omega_P^2 + \frac{3}{5}k^2
\end{aligned} \tag{4.168}$$

4.5.2 Transverse modes

For the transverse excitations, we can similarly try to solve the dispersion relations from the equation

$$(k^0)^2 - k^2 - m^2\left(\frac{k^0}{k}\right)^2 - m^2\left(\frac{k^0}{k}\right)\left[1 - \left(\frac{k^0}{k}\right)^2\right]Q_0\left(\frac{k^0}{k}\right) = 0 \tag{4.169}$$

with a substitution of $x = k^0/k = \omega_T/k$, this is

$$(x^2 - 1) - \frac{m^2}{k^2}\left[x^2 + \frac{x(1-x^2)}{2}\ln\frac{x+1}{x-1}\right] \tag{4.170}$$

In the high momentum limit, $x \approx 1$, the second term in the square bracket

$$\begin{aligned}
\lim_{x\to 1}\frac{x(1-x^2)}{2}\ln\frac{x+1}{x-1} &= \lim_{x\to 1}\frac{x(1+x)(1-x)}{2}\left[\ln(x+1) - \ln(x-1)\right] \\
&= \lim_{x\to 1}\frac{x-1}{\ln(x-1)} \\
&= \lim_{x\to 1}\frac{1}{1/(x-1)} \to 0
\end{aligned} \tag{4.171}$$

goes to zero and thus can be dropped, since the other term x^2 in the bracket is of order 1. Note that $x^2 - 1 = [\omega_T^2 - k^2]/k^2 \sim m^2/k^2$, the two terms in (4.170) are of the same order. So we obtain

$$x^2 - 1 - \frac{m^2}{k^2} = 0 \tag{4.172}$$

and immediately find the dispersion relation

$$\omega_T^2 = k^2 + m^2 \tag{4.173}$$

For small momenta k, using the expansion formula (4.165), we find

$$\begin{aligned} x^2 + \frac{x(1-x^2)}{2} \ln \frac{x+1}{x-1} &= x^2 + x(1-x^2)\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right) \\ &= x^2 + 1 + \frac{1}{3x^2} + \frac{1}{5x^4} - x^2 - \frac{1}{3} - \frac{1}{5x^2} + \cdots \\ &= \frac{2}{3} + \frac{2}{15x^2} \end{aligned} \tag{4.174}$$

Plugging this into (4.170) and replacing $k^0/k = x$, we get

$$\omega_T^4 - (\omega_P^2 + k^2)\omega_T^2 - \frac{1}{5}\omega_P^2 k^2 = 0 \tag{4.175}$$

from which we solve

$$\begin{aligned} \omega_T^2 &= \frac{1}{2}\left[\omega_P^2 + k^2 + \sqrt{(\omega_P^2+k^2)^2 + \frac{4}{5}\omega_P^2 k^2}\right] \\ &= \frac{1}{2}\left\{\omega_P^2 + k^2 + \omega_P^2\left[1 + \frac{1}{2}\left(2 + \frac{4}{5}\right)\frac{k^2}{\omega_P^2}\right]\right\} \\ &= \omega_P^2 + \frac{6}{5}k^2 \end{aligned} \tag{4.176}$$

The exact solutions of (4.161) and (4.170) can of course be solved numerically. The curves of dispersion relations for the longitudinal and transverse photon excitations are shown in Fig.2.

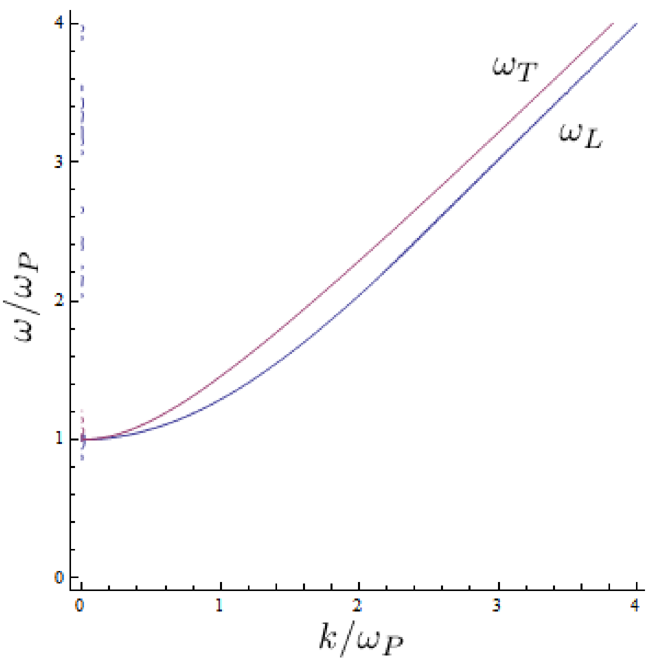


Figure 2: Dispersion relations for photon excitations

For large momenta k, the asymptotic behaviors of longitudinal and transverse excitations are distinct. However, for low momenta $k \to 0$, the frequencies of longitudinal and transverse modes are indistinguishable, $\omega_L(0) = \omega_T(0) = \omega_P$. The intercept of the curve is the plasma frequency, which describes the effective mass of the quasi-particles. The longitudinal excitation is sometimes called a plasmon, and appears in medium only.

4.6 Debye screening

The collective modes and other physical quantities of interest can also be studied by linear response theory.

Suppose we introduce a classical current in a hot QED plasma, which gives a small perturbation to the Hamiltonian

$$V = \int d^3\mathbf{x}^3 j^{\mu}_{cl}(x) A_{\mu}(x) \tag{4.177}$$

By linear response, the small change in the gauge field in the medium is

$$\langle \delta A_{\mu}(x) \rangle = -\int d^4y \theta(x^0 - y^0) \langle [A_{\mu}(x), A_{\nu}(y)] \rangle_{\beta} j^{\nu}_{cl}(x) \tag{4.178}$$

This is basically the retarded correlator. Note that in the absence of $j^{\mu}_{cl}(x)$, the mean field vanishes $\langle A_{\mu}(x) \rangle_{\beta} = 0$, as it is the expectation of a one-point function in a medium. Thus δA_{μ} is basically A_{μ}.

Write down the Fourier transforms

$$D^{\mu\nu}_R(x-y) = \int \frac{d^4K}{(2\pi)^4} \mathrm{e}^{-iK(x-y)} D^{\mu\nu}_R(K) \tag{4.179}$$

$$j^{\mu}_{cl}(x) = \int \frac{d^4P}{(2\pi)^4} \mathrm{e}^{-iPx} j^{\mu}_{cl}(P) \tag{4.180}$$

We also have

$$\begin{aligned} A^{\mu}(x) &= \int d^4y D^{\mu\nu}_R(x-y) j_{cl,\nu}(y) \\ &= \int \frac{d^4K}{(2\pi)^4} \mathrm{e}^{-iKx} D^{\mu\nu}_R(k) j_{cl,\nu}(k) \end{aligned} \tag{4.181}$$

from which we read off

$$A^{\mu}(K) = D^{\mu\nu}_R(K) jcl, \nu(K) = \frac{D^{\mu\nu}_L}{K^2 - F} j_{cl,\nu} + \frac{D^{\mu\nu}_T}{K^2 - G} j_{cl,\nu} \tag{4.182}$$

In components, we have

$$A^0(K) = -\frac{P^{0i}_L j^i}{K^2 - F} j_{cl,\nu} + \frac{P^{00}_L j^0}{K^2 - F} \tag{4.183}$$

To simplify this, we note that

$$P_L^{0i} = \frac{k^0 k^i}{k^2}, \qquad P_L^{00} = -g^{00} + \frac{k^0 k^0}{k^2} \tag{4.184}$$

and also owing to the conservation of currents

$$k_\mu j_{cl}^\mu = k^0 j^0 - k^i j^i = 0 \tag{4.185}$$

we can get the following relation

$$P_L^{0i} j^i = \frac{k^0 k^i j^i}{(k^0)^2 - k^2} = \frac{(k^0)^2 j^0}{(k^0)^2 - k^2} \tag{4.186}$$

Substitute these in (4.183) and we have

$$\begin{aligned} A^0 &= \frac{1}{K^2 - F}\left\{ -\frac{(k^0)^2}{(k^0)^2 - k^2} - 1 + \frac{(k^0)^2}{(k^0)^2 - k^2} \right\} j^0 \\ &= -\frac{\rho_{cl}}{(k^0)^2 - k^2 - F} \end{aligned} \tag{4.187}$$

where we identified the zeroth component of the 4-current as the charge density ρ_{cl}. The i-th component of the gauge vector is

$$\begin{aligned} A^i &= \frac{-P_T^{ij} j^j}{K^2 - G} + \frac{P_L^{i0} j^0 - P_L^{ij} j^j}{K^2 - F} \\ &= -\left\{ \frac{j_{cl,T}^i}{(k^0)^2 - k^2 - G} + \frac{j_{cl,L}^i}{(k^0)^2 - k^2 - F} \right\} \end{aligned} \tag{4.188}$$

From these one can compute measurable quantities, such as the induced electric and magnetic fields in the plasma. For electric fields, one has

$$E^i(x) = F^{i0} = \partial^i A^0(x) - \partial^0 A^i(x) \tag{4.189}$$

$$E^i(K) = -ik^i A^0(K) + ik^0 A^i(K) \tag{4.190}$$

and simliar simple relations hold for the B fields.

Let us consider the special case of a point charge placed in a thermal bath.

$$j^i = 0 \qquad j^0 = \rho(\mathbf{x}) \tag{4.191}$$

Assuming the charge density is independent from time t, then the Fourier transform is

$$\rho(K) = \int d^4x e^{iKx} \rho(\mathbf{x}) = \rho(\mathbf{k}) \cdot 2\pi\delta(k^0) \tag{4.192}$$

Say we have a point charge at the origin

$$\rho(\mathbf{x}) = Q\delta^{(3)}(\mathbf{x}) \tag{4.193}$$

then in the energy-momentum space,

$$\rho(K) = \int d^4x \mathrm{e}^{iKx} Q\delta^{(3)}(\mathbf{x}) = 2\pi Q\delta(k^0) \tag{4.194}$$

We can compute the induced fields

$$\begin{aligned} A^0(x) &= \int \frac{d^4K}{(2\pi)^4} \mathrm{e}^{-iKx} A^0(k) \\ &= \int \frac{d^4K}{(2\pi)^4} \mathrm{e}^{-iKx} \frac{-2\pi Q\delta(k^0)}{(k^0)^2 - k^2 - F(k^0, \mathbf{k})} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \frac{Q}{k^2 + F(0, \mathbf{k})} \end{aligned} \tag{4.195}$$

At $k^0 = 0$, we have from (4.140), $F(0, \mathbf{k}) = 2m^2$, so

$$A^0(\mathbf{x}) = Q \int \frac{dk k^2 d\cos\theta d\phi}{(2\pi)^3} \frac{\mathrm{e}^{ikr\cos\theta}}{k^2 + 2m^2} \tag{4.196}$$

The term $m_D^2 = 2m^2 = e^2T^2/3$ is named as the *Debye mass* squared, for a reason that we will see soon. We can integrate over $d\phi$ and $d\cos\theta$ easily, and note that A^0 actually only depends on the argument $r = |\mathbf{x}|$. We write

$$\begin{aligned} A^0(r) &= Q \int_0^\infty \frac{dk k^2}{(2\pi)^2} \int_{-1}^1 dy \frac{\mathrm{e}^{ikry}}{k^2 + m_D^2} \\ &= \frac{Q}{(2\pi)^2} \int_0^\infty dk \frac{k}{k^2 + m_D^2} \frac{\mathrm{e}^{ikr} - \mathrm{e}^{-ikr}}{ir} \\ &= \frac{Q}{(2\pi)^2} \frac{1}{ir} \int_0^\infty dk \frac{k}{k^2 + m_D^2} \mathrm{e}^{ikr} \end{aligned} \tag{4.197}$$

where in the last step we used the property that the integrand is even in k.

There are two poles in the complex k plane, at $k = \pm im_D$ respectively. But since r is positive definite, the contour should be closed as $k \to i\infty$, so the contour is anti-clockwise in the upper half complex plane, picking up a pole at $k = im_D$ only.

$$\begin{aligned} A^0(r) &= \frac{Q}{(2\pi)^2} \frac{2\pi i}{ir} \frac{im_D}{2im_D} \mathrm{e}^{-m_D r} \\ &= \frac{Q}{4\pi r} \mathrm{e}^{-m_D r} \\ &= \frac{Q}{4\pi r} \mathrm{e}^{-r/r_D} \end{aligned} \tag{4.198}$$

If $m_D = 0$, $A^0(r)$ is exactly the vacuum potential of a point charge, known as the Coulomb potential. In a medium, the potential is screened by the medium, and r_D is the distance at which it falls to e^{-1} of the Coulomb potential. This characteristic length scale is called the *Debye radius*.

4.7 Electron self energy

Due to the interactions with photons, the electron propagator should also be modified. The full electron propagator is of the form

$$S(P) = \frac{1}{\not{P} - m_e + \Sigma} \tag{4.199}$$

where $\Sigma(P)$ is the electron self energy. We want to study this at the first order. The corresponding one loop diagram is Fig.3.

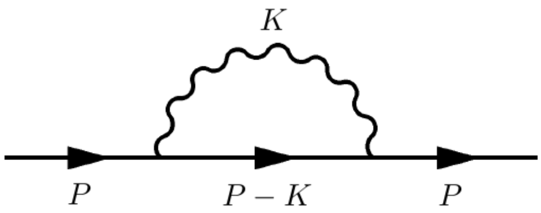


Figure 3: One-loop electron self energy

Using the Feynman rules to write

$$\Sigma(P) = -\frac{e^2}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma^\mu(\not{K} - \not{P})\gamma_\mu \Delta(K)\tilde{\Delta}(P-K) \tag{4.200}$$

We again limit ourselves to the HTL corrections only, where electron mass m_e and external momenta P can be ignored with respect to loop momenta K. With these simplifications, the electron self energy becomes

$$\Sigma(P) = -\frac{2e^2}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\not{K}\Delta(K)\tilde{\Delta}(P-K) \tag{4.201}$$

The procedures to evaluate the frequency sum is studies in detail in the appendix. We will skip the details of calculations and simply quote the results

$$\Sigma(P) = 2e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\not{K}\frac{s_1 s_2}{4E_k E_{p-k}}\frac{1 + n(s_1 E_k) - \tilde{n}(s_2 E_{p-k})}{i\omega - s_1 E_k - s_2 E_{p-k}} \tag{4.202}$$

Carrying out the integration over $d\phi$ and dk, it can be found that the final result can be nicely written as

$$\Sigma(P) = m_f^2 \int \frac{d\Omega}{4\pi}\frac{\hat{\not{K}}}{P\cdot\hat{K}} \tag{4.203}$$

where $m_f^2 = e^2T^2/8$ is identified as the *electron thermal mass* squared, and $\hat{K} = (-i, \hat{\mathbf{k}})$ is a light-like vector.

Taking the angular integrations, the final results in components can be shown to be

$$\Sigma^0 = m_f^2 \frac{2}{p} Q_0\left(\frac{p^0}{p}\right) = \frac{e^2T^2}{8}\frac{2}{p}\ln\frac{p^0+p}{p^0-p} \tag{4.204}$$

$$\mathbf{\Sigma} = m_f^2 \frac{\mathbf{p}}{p^2}\left[1 - \frac{p^0}{2p} Q_0\left(\frac{p^0}{p}\right)\right] = \frac{e^2T^2}{8}\frac{\mathbf{p}}{p^2}\left[1 - \frac{p^0}{2p}\ln\frac{p^0+p}{p^0-p}\right] \tag{4.205}$$

The dispersion relation of the electron collective modes are determined by the poles of the propagator. The defining equation is

$$(P+\Sigma)^2 = (p^0 + \Sigma^0)^2 - (\mathbf{p} + \mathbf{\Sigma})^2 = 0 \tag{4.206}$$

which has two solutions, namely $\omega_\pm(p)$. We can study the asymptotic behaviors of the solutions in the high and low momentum regime. In the high momentum limit, this is

$$\omega_+^2 = p^2 + m_f^2 \tag{4.207}$$

$$\omega_-^2 = p^2 + 4p^2 \exp\left(-\frac{4p^2}{m_f^2}\right) \tag{4.208}$$

while in the low momentum limit both of the solutions behave as

$$\omega_\pm = m_f \pm \frac{1}{3}p \tag{4.209}$$

It is also interesting to look at the asymptotic behaviors of the electron propagator in the two limits. In the high momentum limit,

$$S(\omega \to \omega_+) \approx \frac{1}{2}\frac{\gamma^0 - \mathbf{p}\cdot\gamma}{\omega - \omega_+} \tag{4.210}$$

$$S(\omega \to \omega_-) \approx \frac{2p^2}{m_f^2}\exp\left(-\frac{4p^2}{m_f^2}\right)\frac{\gamma^0 + \mathbf{p}\cdot\gamma}{\omega - \omega_-} \tag{4.211}$$

and in the low momentum limit,

$$S(\omega \to \omega_+) \approx \frac{4}{3}\frac{\gamma^0 - \mathbf{p}\cdot\gamma}{\omega - \omega_+} \tag{4.212}$$

$$S(\omega \to \omega_-) \approx \frac{4}{3}\frac{\gamma^0 - \mathbf{p}\cdot\gamma}{\omega - \omega_-} \tag{4.213}$$

from which we see that the ω_+ mode has the same relation between helicity and chirality, while the ω_- mode has the opposite relation. We may also note that within the high momentum limit, the ω_+ excitation has exactly the same dispersion relation as real electrons, so we conclude that the ω_+ mode is the modification to real electrons at finite temperature. The ω_- mode is a true collective mode absent at zero temperature, and is sometimes called a plasmino. This excitation mode decouples from the plasma at sufficiently high energies.

In the limit of $p \to 0$, $\omega_\pm$ modes are equally important. The physical picture of this behavior is that when the electrons are at rest, we cannot tell from different polarization states.

4.8 Quantum chromodynamics

The structure of the QED and QCD model are very similar, so we might generalize our results from the studies of the QED theory to the QCD case. However, it turns out that QCD is a much more complicated theory. The sophistication of the QCD lies in that the gauge group under which quarks transform is the non-Abelian $SU(3)$ group. The gauge fields in QCD, also known as the gluon fields, are analogous to photon fields in QED theory, but they carry an additional group index. We write $A_\mu = A_\mu^a t^a$, where the t^a's are the group generators. Since the group generators do not commute with each other. When trying to write down a field strength term for the Lagrangian, we will get cubic and quartic gluon self interacting terms, which give rise to new types of vertices for Feynman diagrams. The ghost fields also play an important role in the evaluation of Feynman diagrams.

We can study the collective excitations in a hot QCD plasma from the poles of the full gluon and quark propagators. The one-loop evaluation of the gluon self energy now comes from four diagrams.

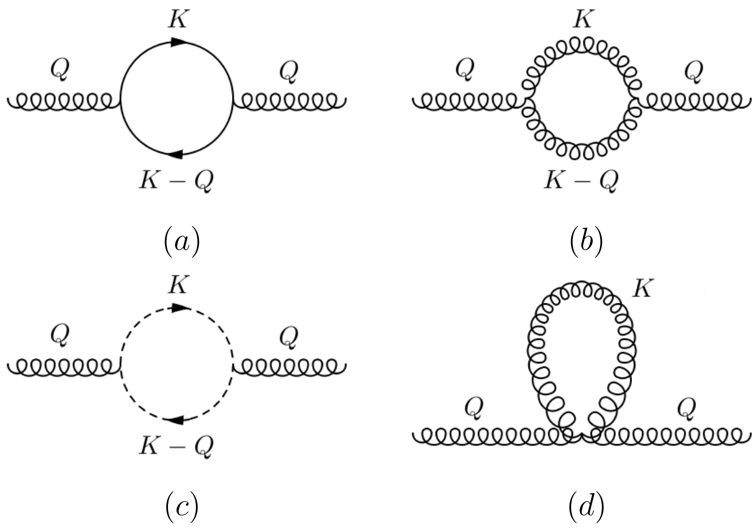


Figure 4: One-loop gluon self energy

The first diagram is similar to the photon self energy, but the rest three diagrams are typical of QCD. Using the corresponding Feynman rules and also taking the HTL approximation, we have

$$\Pi^{(a)}_{\mu\nu} = \frac{1}{2} g^2 N_f \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} (8K_\mu K_\nu - 4K^2 \delta_{\mu\nu}) \tilde{\Delta}(K) \tilde{\Delta}(Q-K) \quad (4.214)$$

$$\Pi^{(b)}_{\mu\nu} = -g^2 C_A \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Delta(K)\Delta(Q-K) \quad (4.215)$$

$$\Pi^{(c)}_{\mu\nu} = g^2 C_A \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} K_\mu K_\nu \Delta(K)\Delta(Q-K) \quad (4.216)$$

$$\Pi^{(d)}_{\mu\nu} = 3C_A g^2 \delta_{\mu\nu} \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} (K^2 \delta_{\mu\nu} + 5K_\mu K_\nu)\Delta(K) \quad (4.217)$$

where g is the QCD coupling constant, $C_A(=3)$ is the group factor, $N_f(=3)$ is the number of quark flavors.

Although the appearance of last three expressions are somehow unfortunate, however, in the HTL limit one might find the contributions to the gluon self energy from all the one-loop diagrams can be simplified to be a compact expression

$$\Pi_{\mu\nu} = -g^2 (C_A + \frac{1}{2} N_f) \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} (4K_\mu K_\nu - 2K^2 \delta_{\mu\nu}) \Delta(K)\Delta(Q-K) \quad (4.218)$$

Note that the frequency sum and integral we need to evaluate has already been done in our discussions for the QED case. To see this more clearly, let us go back to the expression for $\Pi_{\mu\nu}$. Carry out the trace with (4.130) and take the HTL limit, i.e., ignore the external momenta in comparison with loop momenta, we have

$$\Pi_{\mu\nu} = \frac{e^2}{\beta} \sum_n (8K_\mu K_\nu - 4K^2 \delta_{\mu\nu}) \tilde{\Delta}(K) \tilde{\Delta}(Q-K) \quad (4.219)$$

The only difference between (4.218) and (4.218) is the overall factor. So we may simply borrow the results. We speculate that at finite temperature a thermal gluon mass given by

$$m_g^2 = \frac{1}{6} g^2 T^2 (C_A + \frac{1}{2} N_f) \quad (4.220)$$

will be generated in the heat bath.

The evaluation of quark self energy at one-loop order is even simpler, as there is only one relevant diagram which is just the same as Fig.3. The result is

$$\Sigma(P) = m_f^2 \int \frac{d\Omega}{4\pi} \frac{\hat{\not{K}}}{P \cdot \hat{K}} \quad (4.221)$$

with the quark thermal mass given by

$$m_f^2 = \frac{1}{8} g^2 T^2 C_f \quad (4.222)$$

where C_f is a group factor.

Therefore, we can learn about the collective modes in a QCD plasma from our knowledge of QED plasma excitations with the replacements

$$e^2T^2 \to g^2T^2(C_A + \frac{1}{2}N_f) \qquad \text{for gluons} \qquad (4.223)$$

$$e^2T^2 \to g^2T^2C_f \qquad \text{for quarks} \qquad (4.224)$$

5 Hard Thermal Loops and Resummation Rules

5.1 The breakdown of a naive perturbative theory

5.1.1 Scalar theories

In conventional quantum field theories, we perturbatively expand the theory in terms of the dimensionless coupling constant order by order. However, in thermal field theories, some Feynman diagrams of superficially higher orders in the coupling constant might have the same magnitude as lower order diagrams owing to the contributions from the so-called hard thermal loops corrections.

These problematic issues are caused by the infrared divergences. We may revisit the massless scalar ϕ^4 model to see how a naive perturbation series could break down. We have studied the corrections to the partition function to the first order. However, if we want to go further, say compute the partition function at the second order, we will encounter divergences from loop integrals. One of the diagrams contributing at the second order is $\bigcirc\!\bigcirc\!\bigcirc$. Using the Feynman rules, this looks like

$$\bigcirc\!\bigcirc\!\bigcirc \sim \lambda^2 \left[\frac{1}{\beta}\sum_m \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_m^2+\mathbf{k}^2}\right]^2 \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{(\omega_n^2+\mathbf{p}^2)^2} \tag{5.1}$$

Let us look at the second integral piece. At a low momentum limit, it is divergent for the $n=0$ mode, the integral behaves as

$$\int \frac{d^3\mathbf{p}}{(\omega_n^2+\mathbf{p}^2)^2}\Big|_{n=0} \sim \int \frac{dp}{p^2} \tag{5.2}$$

We say this is infrared divergent as $p \to 0$. This divergence is apparently different from the ultraviolet divergences at high momenta. For an expansion of higher order in λ, there are more possible infrared divergent diagrams. But one may find that at each order, the most infrared divergent diagram takes the form of Fig.5.

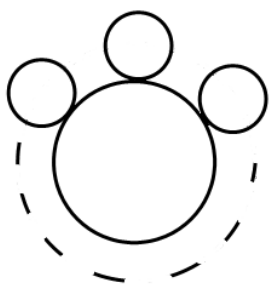
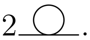


Figure 5: Ring diagrams

To study this sort of ring diagrams more closely, let us work out the combinatoric symmetry factor of such a diagram with N loops as follows. There are $(N-1)!/2$

ways of ordering the vertices. There are $4 \times 3 = 12$ ways to connect different lines at each vertex, but this factor is already included in the diagrammatic expression $\Pi = -12$. There is yet a factor of $1/N!$ from the Taylor expansion. We can further sum over all these ring diagrams to miraculously get a finite result. The summation over all the ring diagrams from $N = 2$ to infinity reads

$$\begin{aligned}
&\frac{1}{2}\beta V \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{N=2}^{\infty} \frac{1}{N}\left(-\Pi_1 D_0(\omega_n, \mathbf{p})\right)^N \\
=&-\frac{V}{2}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\ln(1+\Pi_1 D_0) - \Pi_1 D_0\right]
\end{aligned} \tag{5.3}$$

Replacing the leading order correction to the self energy $\Pi_1 = \lambda T^2$, this becomes

$$-\frac{V}{2}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\ln(1+\frac{\lambda T^2}{\omega_n^2+\mathbf{p}^2}) - \frac{\lambda T^2}{\omega_n^2+\mathbf{p}^2}\right] \tag{5.4}$$

Taking the $n = 0$ mode to evaluate the dominating contribution

$$\begin{aligned}
&-\frac{V}{2}\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\ln(1+\frac{\lambda T^2}{\mathbf{p}^2}) - \frac{\lambda T^2}{\mathbf{p}^2}\right] \\
=&-\frac{V}{2}\frac{\lambda^{3/2}T^3}{4\pi^2}\int dx\sqrt{x}\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\right] \\
=&\frac{\lambda^{3/2}}{12\pi^2}VT^3
\end{aligned} \tag{5.5}$$

We see that this gives a correction to the partition function of order $\lambda^{3/2}$, which is obviously not expected from a naive perturbation expansion in a progressive order in λ.

The occurrence of the infrared divergence is due to the fact that the massless scalar particles acquire a thermal mass in a thermal background. If we examine the source of the scalar thermal mass more carefully, by looking at the expression (**??**) we obtained in the one-loop evaluation for scalar self energy which is

$$\Pi_T = 12\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\omega_p}\frac{1}{\mathrm{e}^{\beta\omega_p}-1} \tag{5.6}$$

We see the leading contribution to the integral comes from hard momenta $p \sim T$, where $\mathrm{e}^{\beta\omega_p} - 1 \to 0$. In other words, it is from the hard loops that the thermal mass of order λT is generated. For the soft momenta $p \sim \sqrt{\lambda}T$, We might expect their contribution to the thermal mass is also of order $\lambda^{3/2}T$, from the speculation that the contribution to the partition function from the ring diagrams, which are the dominating contributions to the infrared divergences for soft momenta at all orders, is of order $\lambda^{3/2}$.

Recall that the full propagator of the interacting theory is expressed in terms of a bare propagator with a correction from the self-energies of order λT^2

$$D^{-1}(\omega_n, \mathbf{p}) = \omega_n^2 + \mathbf{p}^2 + \Pi(\omega_n, \mathbf{p}) \approx \omega_n^2 + \mathbf{p}^2 + \lambda T^2 \tag{5.7}$$

For a particle of soft momentum of order $\sqrt{\lambda}T$, the inverse of its free propagator is of the same magnitude as the thermal mass. But we are expanding the perturbative series with free scalar propagators without considering the thermally generated mass, which may cause problems.

These give us a hint that a naive perturbation theory breaks down for soft momenta, which should be resummed. However, it can be found that the only HTL contribution to λT^2 is the 2-point correlators, so it is adequate to use an effective propagator

$$D^*(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + \lambda T^2} \tag{5.8}$$

to obtain an improved perturbative expansion.

5.1.2 Gauge theories

The discussions of scalar field theories give us enough information on the gauge theories. For gauge theories, it is also necessary to resum all the possible HTL contributions.

However, the structure of gauge theories is much more complicated than scalar theories. Unlike the latter, where the self energies are dependent on temperature only, the gauge theory self energies are dependent on temperature as well as energy and momentum.

One might try to examine what sort of hard thermal loops result into a leading T^2 behavior in QED or QCD. The complete discussions are very involved, However, attributed to Braaten and Pisarski [7] [8], one can use a set of power counting rules to understand the superficial magnitude of the HTL contributions. For a proper vertex with N legs, without taking into account the coupling constant g, the order of the contribution from a one-loop HTL can be counted by the following rules first established by Braaten and Pisarski.

1. The integration over loop momentum $\int d^3k$ is of magnitude T^3.
2. There is a factor of T^{-1} for the first propagator from the frequency sum. Each additional propagator will contribute a factor of $(pT)^{-1}$.
3. Each power of k from the numerator, arising from propagators or vertices, shall be replaced by T
4. For propagators which represent fields with the same statistics, there is a statistical factor of pT^{-1} for the cancellation of distribution factors.

To utilize these rules, for example, for a $(N-2)$-photon-two-electron vertex, one has

$$e^N T^3 (pT)^{-N} (pT^{-1})^1 T^N = g^N T^2 p^{-N+1} \tag{5.9}$$

In the case of a three point electron-photon vertex, this is $e^3 T^2 p^{-2}$. For soft momentum $p \sim eT$, the HTL correction given by the power counting rules is thus e, which is of the same order as a bare vertex. This again gives us a sign that the perturbation expansion breaks down for soft momenta.

5.2 The effective perturbative theory

An effective theory can be formalized by resumming the hard thermal loop contributions. In this formalism, bare vertices and bare propagators shall be replaced by the effective vertices and effective propagators. We have explained for the scalar field theories that one can improve the perturbation expansion by simply replacing the bare propagator with an effective propagator (5.8). For QED one might write down the effective theory by starting with effective Lagrangians [9]. The effective Lagrangian for photon hard thermal loops is

$$\mathcal{L}_\gamma = -\frac{m^2}{2} \int \frac{d\Omega}{4\pi} F_{\mu\alpha} \frac{\hat{K}_\alpha \hat{K}_\beta}{(\hat{K} \cdot D)^2} F^\mu_\beta \tag{5.10}$$

where $F_{\mu\nu}$ is the field strength defined before, and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. The generalization to the QCD case is straightforward. Denoting the group generators by t^a, one only needs to replace the Abelian field strength by the non-Abelian field strength $F_{\mu\nu} = F^a_{\mu\nu} t^a$ and take an overall trace over the group indices.

The effective Lagrangian for fermionic hard thermal loops is

$$\mathcal{L}_f = -m_f^2 \bar{\psi}(x) \int \frac{d\Omega}{4\pi} \frac{\hat{\not{K}}}{\hat{K} \cdot D} \psi(x) \tag{5.11}$$

It is notable that the effective Lagrangians are manifestly Lorentz covariant.

All the n-point functions of our interest can be generated from the effective Lagrangians (5.10) and (5.11). For example, one obtains the electron-photon vertex by computing $\delta\mathcal{L}_f / \delta\bar{\psi}\delta\psi\delta A_\mu$. The result is

$$\Gamma_\mu(P_1, P_2) = -m_f^2 \int \frac{d\Omega}{4\pi} \frac{\hat{K}_\mu \hat{\not{K}}}{(P_1 \cdot \hat{K})(P_2 \cdot \hat{K})} \tag{5.12}$$

By using the expression (4.203) for one-loop electron self energy in the HTL limit, one can check that the expression for electron photon vertex satisfies the tree-level

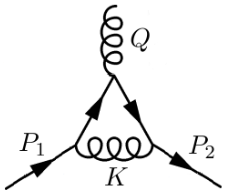


Figure 6: One loop electron photon vertex

Ward identity

$$(P_1 - P_2)^\mu \Gamma_\mu(P_1, P_2) = \Sigma(P_1) - \Sigma(P_2) \tag{5.13}$$

This should be expected because the hard thermal loops give a contribution to the leading order at high temperatures, where Lorentz covariance still requires the Ward identity must be obeyed.

The same expression as (5.12) can also be derived from an explicit calculation of the vertex function at one loop order within HTL approximation. From the Feynman rules, we have

$$e\Gamma_\mu(P_1, P_2) = \frac{e^3}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\gamma^\alpha(\not{K} - \not{P}_2)\gamma_\mu(\not{K} - \not{P}_1)\gamma_\alpha\right] \Delta(K)\tilde{\Delta}(P_2 - K)\tilde{\Delta}(P_1 - K) \tag{5.14}$$

which can be simplified to be

$$\Gamma_\mu(P_1, P_2) = -4e^2 \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} K_\mu \not{K} \Delta(K)\tilde{\Delta}(P_2 - K)\tilde{\Delta}(P_1 - K) \tag{5.15}$$

under HTL approximations. (5.12) can be obtained by taking the frequency sum and decoupled integrations over dk and $d\phi$. But in order to check the Ward identity, there is no need to do the tedious calculations. Note that within the HTL limit we have

$$(P_1 - P_2)K \approx \frac{1}{2}(P_2 - K)^2 - \frac{1}{2}(P_1 - K)^2 \tag{5.16}$$

Multiply both sides of (5.15) by $(P_1 - P_2)$ and compare with (**??**), we find

$$\begin{aligned}(P_1 - P_2)\Gamma_\mu(P_1, P_2) &= \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \not{K} \left[\Delta(K)\tilde{\Delta}(P_1 - K) - \Delta(K)\tilde{\Delta}(P_2 - K)\right] \\ &= \Sigma(P_1) - \Sigma(P_2)\end{aligned} \tag{5.17}$$

The check of the validation of Ward identities shows the correctness of the effective theory, from which one could further establish a consistent perturbation expansion. The poor convergence due to soft momenta is improved through this reorganization of expansion series. The resummation procedures have enabled physicts to calculate the thermodynamicals to higher orders. The partition functions have been calculated explicitly to the order of g^6 [10] and of $g^8 \ln g$ [11] for a massless $g^2\phi^4$-theory, of e^5 for QED [12], and of $g^6 \ln g$ for QCD [13] [14]. However, there are still infrared divergence problems from the momenta of order g^2T unsolved if one wants to do calculations to higher orders.

A Frequency Sums

We often encounter frequency sums over Matsubara frequencies, so it is necessary to develop some formal methods to carry out these sums.

For the bosonic Matsubara frequencies $\omega_n = 2\pi n/\beta$, we write down schematically a general form of the frequency sum

$$\frac{1}{\beta}\sum_n f\left(i\omega_n = i\frac{2\pi n}{\beta}\right) \tag{A.1}$$

Note that the hyperbolic cotangent function $\coth(\beta p^0/2)$ exactly produces a collection of poles at $p^0 = 2\pi n i/\beta = i\omega_n$, so the frequency sum could be conveniently expressed as a contour integral in the complex p^0 plane as

$$\frac{1}{\beta}\sum_n f\left(p^0 = i\omega_n\right) = \frac{1}{\beta}\int_{C_1}\frac{dp^0}{2\pi i}f(p^0)\frac{\beta}{2}\coth\left(\frac{\beta p^0}{2}\right) \tag{A.2}$$

The contour C_1 is a strip with infinitesimal width around the imaginary axis as illustrated in Fig 7. However we can deform the contour to C_2, which covers the whole complex plane except the imaginary axis. The hyperbolic cotangent function does not produce poles off the imaginary axis, so all the poles within the contour C_2 are only relevant to the explicit expression of the function $f(p^0)$.

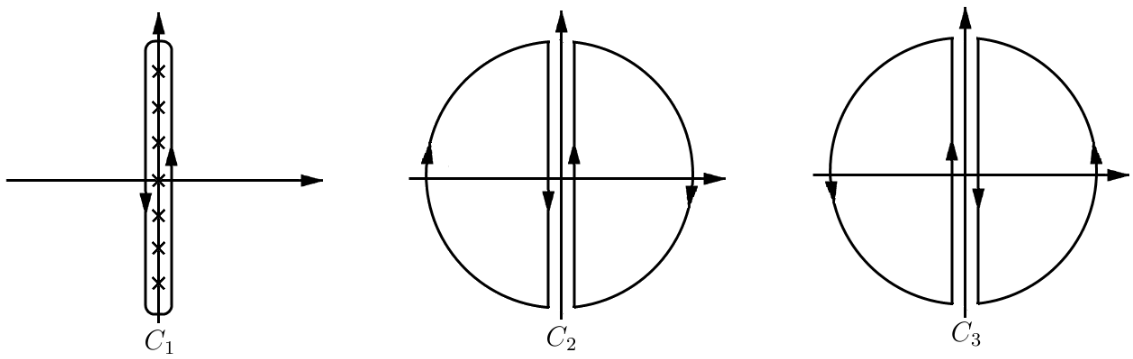


Figure 7: Integration contours

We can further change the contour into C_3 which runs in the opposite direction of C_2 so that it goes counter clockwise and is consistent with our usual convention of contour choices. We shall get a minus sign for doing this, so we have

$$\frac{1}{\beta}\sum_n f\left(p^0 = i\omega_n = i\frac{2\pi n}{\beta}\right) = -\frac{1}{\beta}\int_{C_3}\frac{dp^0}{2\pi i}f(p^0)\frac{\beta}{2}\coth\left(\frac{\beta p^0}{2}\right) \tag{A.3}$$

Then we pick up all the poles produced by $f(p^0)$ and calculate their residuals to obtain an evaluation of the original frequency sum.

For example, we want to evaluate in (3.15)

$$\sum_n \frac{1}{(\beta\omega_n)^2+x^2} = \sum_n \frac{1}{(2\pi n)^2+x^2} = \sum_n \frac{1}{-(\beta p^0)^2+x^2}\bigg|_{p^0=i\frac{2\pi n}{\beta}} \tag{A.4}$$

With the tricks introduced above, we have

$$\int_{C_1} \frac{dp^0}{2\pi i}\frac{1}{-(\beta p^0)^2+x^2}\frac{\beta}{2}\coth\left(\frac{\beta p^0}{2}\right) = \int_{C_3} \frac{dp^0}{2\pi i}\frac{1}{(\beta p^0)^2-x^2}\frac{\beta}{2}\coth\left(\frac{\beta p^0}{2}\right) \tag{A.5}$$

The two poles $p^0 = \pm x/\beta$ give rise to a sum of residuals

$$\frac{1}{2\beta x}\frac{\beta}{2}\coth\frac{x}{2} + \frac{1}{-2\beta x}\frac{\beta}{2}\coth\frac{-x}{2} = \frac{1}{2x}\coth\frac{x}{2} = \frac{1}{2x}\left(1+\frac{2}{\mathrm{e}^x-1}\right) \tag{A.6}$$

Hence we obtain

$$\sum_n \frac{1}{(2\pi n)^2+x^2} = \frac{1}{2x}\left(1+\frac{2}{\mathrm{e}^x-1}\right) \tag{A.7}$$

For fermionic Matsubara frequencies, the ideas are very similar but we should instead use a hyperbolic tangent function $\tanh(\beta p^0/2)$. This generates exactly the poles at $p^0 = 2\pi(n+1)i/\beta = i\omega_n$ as required. We have

$$\frac{1}{\beta}\sum_n f\left(p^0 = i\omega_n = i\frac{2\pi(n+1)}{\beta}\right) = -\frac{1}{\beta}\int_{C_3}\frac{dp^0}{2\pi i} f(p^0)\frac{\beta}{2}\tanh\left(\frac{\beta p^0}{2}\right) \tag{A.8}$$

A simple example can be illustrated with the evaluation of (4.32). We compute

$$\begin{aligned}
\sum_n \frac{1}{\beta^2\omega_n^2+x^2} &= \sum_n \frac{1}{-(\beta p^0)^2+x^2}\bigg|_{p^0=i\frac{2\pi(n+1)}{\beta}} \\
&= \int_{C_3}\frac{dp^0}{2\pi i}\frac{1}{(\beta p^0)^2-x^2}\frac{\beta}{2}\tanh\left(\frac{\beta p^0}{2}\right) \\
&= \frac{1}{2\beta x}\frac{\beta}{2}\tanh\frac{x}{2}\times 2 \\
&= \frac{1}{2x}\left(1-\frac{2}{\mathrm{e}^x+1}\right)
\end{aligned} \tag{A.9}$$

We also encounter more complicated frequency sums which contain several propagators during the evaluation of some loop integrals, so it will be helpful to learn how to do them. We can of course perform the sums by evaluation the corresponding contour integral directly. There is also another very useful trick which we will come to soon. By giving another explicit example, we first illustrate again the contour integral method, and then evaluate it in a different approach.

When evaluating the contributions from the loop diagrams, one may encounter a sum involving two bosonic propagators of the following form

$$\begin{aligned}
S(i\omega_m,\mathbf{p}) &= \frac{1}{\beta}\sum_n \Delta(i\omega_n,\mathbf{k})\Delta(i(\omega_n-\omega_m),\mathbf{k}-\mathbf{p}) \\
&= \frac{1}{\beta}\sum_n \frac{1}{\omega_n^2+E_1^2}\frac{1}{(\omega_n-\omega_m)^2+E_2^2}
\end{aligned} \tag{A.10}$$

where $E_1 = E_{\mathbf{k}}$, $E_2 = E_{\mathbf{p}-\mathbf{k}}$. Using the contour integral evaluation method, we rewrite this as

$$-\frac{1}{\beta}\int_{C_3}\frac{dk^0}{2\pi i}\frac{1}{(k^0)^2-E_1^2}\frac{1}{(k^0-i\omega_m)^2-E_2^2}\frac{\beta}{2}\coth\left(\frac{\beta k^0}{2}\right) \tag{A.11}$$

Within the contour C_3 on the complex k^0 plane, we pick up four poles, namely

$$k^0 = \pm E_1, i\omega_m \pm E_2 \tag{A.12}$$

Note that the hyperbolic cotangent functions have the following properties

$$\begin{aligned}
\coth\frac{\beta k^0}{2} &= \frac{\mathrm{e}^{\beta k^0/2}+\mathrm{e}^{-\beta k^0/2}}{\mathrm{e}^{\beta k^0/2}-\mathrm{e}^{-\beta k^0/2}} \\
&= \frac{\mathrm{e}^{\beta k^0}+1}{\mathrm{e}^{\beta k^0}-1} = 1+\frac{2}{\mathrm{e}^{\beta k^0}-1} \qquad \text{for } k^0>0 \qquad (\text{A.13}) \\
&= \frac{1+\mathrm{e}^{-\beta k^0}}{1-\mathrm{e}^{-\beta k^0}} = -1-\frac{2}{\mathrm{e}^{-\beta k^0}-1} \qquad \text{for } k^0<0 \qquad (\text{A.14})
\end{aligned}$$

The first two poles correspond to residuals

$$\begin{aligned}
k^0 = E_1 \quad &\to \quad -\frac{1}{2}\frac{1}{2E_1}\frac{1}{(E_1-i\omega_m)^2-E_2^2}\left(1+\frac{2}{\mathrm{e}^{\beta E_1}-1}\right) \\
&= \frac{1}{2}\frac{1+2n_1}{2E_1\cdot 2E_2}\left[\frac{1}{i\omega_m-E_1-E_2}-\frac{1}{i\omega_m-E_1+E_2}\right] \\
k^0 = -E_1 \quad &\to \quad -\frac{1}{2}\frac{-1}{2E_1}\frac{1}{(-E_1-i\omega_m)^2-E_2^2}\left(-1-\frac{2}{\mathrm{e}^{\beta E_1}-1}\right) \\
&= \frac{1}{2}\frac{1+2n_1}{2E_1\cdot 2E_2}\left[\frac{1}{i\omega_m+E_1-E_2}-\frac{1}{i\omega_m+E_1+E_2}\right]
\end{aligned}$$

while the other two poles correspond to residuals

$$\begin{aligned}
k^0 = i\omega_m+E_2 \quad &\to \quad -\frac{1}{2}\frac{1}{2E_2}\frac{1}{(i\omega_m+E_2)^2-E_1^2}\left(1+\frac{2}{\mathrm{e}^{\beta(i\omega_m+E_2)}-1}\right) \\
&= \frac{1}{2}\frac{1+2n_2}{2E_1\cdot 2E_2}\left[\frac{1}{i\omega_m+E_2-E_1}-\frac{1}{i\omega_m+E_2+E_1}\right] \\
k^0 = i\omega_m-E_2 \quad &\to \quad \to -\frac{1}{2}\frac{-1}{2E_2}\frac{1}{(i\omega_m-E_2)^2-E_1^2}\left(-1-\frac{2}{\mathrm{e}^{-\beta(i\omega_m-E_2)}-1}\right) \\
&= \frac{1}{2}\frac{1+2n_2}{2E_1\cdot 2E_2}\left[\frac{1}{i\omega_m-E_2-E_1}-\frac{1}{i\omega_m-E_2+E_1}\right]
\end{aligned}$$

When deriving these two expressions, we have assumed that the external momentum P also corresponds to a bosonic particle, so the expressions can be simplified by $\exp(i\beta\omega_n) = 1$. If the external momentum corresponds to a fermionic particle, then in this case we should use $\exp(i\beta\omega_n) = -1$ and therefore several changes of signs should be taken care of.

Picking up the residuals of all the four poles, the value of the frequency sum is

$$S(i\omega_m, \mathbf{p}) = \frac{1}{4E_1E_2}\Bigg[(1+n_1+n_2)\left(\frac{1}{i\omega_m - E_1 - E_2} - \frac{1}{i\omega_m + E_1 + E_2}\right) \\ + (n_2 - n_1)\left(\frac{1}{i\omega_m - E_1 + E_2} - \frac{1}{i\omega_m + E_1 - E_2}\right)\Bigg] \tag{A.15}$$

Apart from using the contour integral procedures to do the frequency sums, there is another approach which might be more convenient in some situations. One can first Fourier transform the Euclidean propagators

$$\Delta(\tau, \mathbf{k}) = \frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n\tau}\Delta(i\omega_n, \mathbf{k}) \tag{A.16}$$

This sum can be easily carried out by using the contour integral method as before. The result is

$$\Delta(\tau, \mathbf{k}) = \frac{1}{2E_{\mathbf{k}}}\left[(1+n(E_{\mathbf{k}}))\mathrm{e}^{-E_{\mathbf{k}}\tau} + n(E_{\mathbf{k}})\mathrm{e}^{E_{\mathbf{k}}\tau}\right] \tag{A.17}$$

Using the property of Bose distribution factor

$$n(-E) = \frac{1}{\mathrm{e}^{-\beta E} - 1} = -1 - \frac{1}{\mathrm{e}^{\beta E} - 1} = -1 - n(E) \tag{A.18}$$

We rewrite

$$\Delta(\tau, \mathbf{k}) = \sum_{s=\pm}\frac{-s}{2E_{\mathbf{k}}} n(-sE_{\mathbf{k}})\mathrm{e}^{-sE_{\mathbf{k}}\tau} \tag{A.19}$$

This is our central formula to evaluate frequency sums in the new approach.

Going back to the frequency-momentum space, the original Euclidean propagators can be rewritten as

$$\begin{aligned} \Delta(\tau, \mathbf{k}) &= \int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau}\Delta(\tau, \mathbf{k}) \\ &= \sum_s \int_0^\beta d\tau \mathrm{e}^{(i\omega_n - sE_{\mathbf{k}})\tau} \times \frac{-s}{2E_{\mathbf{k}}} n(-sE_{\mathbf{k}}) \\ &= \sum_s \frac{\mathrm{e}^{i\omega_n\beta}\mathrm{e}^{-sE_{\mathbf{k}}\beta} - 1}{i\omega_n - sE_{\mathbf{k}}} \frac{-s}{2E_{\mathbf{k}}} \frac{1}{\mathrm{e}^{-sE_{\mathbf{k}}\beta} - 1} \\ &= \sum_s \frac{-s}{2E_{\mathbf{k}}} \frac{1}{i\omega_n - sE_{\mathbf{k}}} \end{aligned} \tag{A.20}$$

The evaluation of the frequency sums over a product of propagators will become easier with this trick. We may show the convenience of this trick by revisiting the evaluation of $S(i\omega_m, \mathbf{p})$ defined in (A.10). Using (A.19) to write

$$S(i\omega_m, \mathbf{p}) = \int_0^\beta d\tau \mathrm{e}^{i\omega_m\tau} \sum_{s_1, s_2} \frac{-s_1}{2E_1} n(-s_1E_1)\mathrm{e}^{-s_1E_1\tau} \frac{-s_2}{2E_2} n(-s_2E_2)\mathrm{e}^{-s_2E_2\tau} \tag{A.21}$$

Taking the integration over $d\tau$ and simplifying the expression, we get

$$\begin{aligned} S(i\omega_m, \mathbf{p}) &= \sum_{s_1,s_2} \frac{s_1 s_2}{4E_1E_2} \frac{1 + n(-s_1E_1) + n(-s_2E_2)}{i\omega_m - s_1E_1 - s_2E_2} \\ &= -\sum_{s_1,s_2} \frac{s_1 s_2}{4E_1E_2} \frac{1 + n(s_1E_1) + n(s_2E_2)}{i\omega_m - s_1E_1 - s_2E_2} \end{aligned} \tag{A.22}$$

The same expression (A.15) which we derived in a different procedure is recovered easily, as it must be.

This method can also be applied to evaluate the frequency sum of fermionic propagators. The ideas are basically the same, so we only briefly give the important results.

The Fourier transform of a Euclidean propagator for fermions

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + E_{\mathbf{p}}^2} = \frac{1}{(2\pi(n+1)/\beta)^2 + E_{\mathbf{p}}^2} \tag{A.23}$$

into the mixed space of imaginary time and momentum is

$$\tilde{\Delta}(\tau, \mathbf{p}) = \sum_{s=\pm} \frac{s}{2E_{\mathbf{p}}} \tilde{n}(-sE_{\mathbf{p}}) \mathrm{e}^{-sE_{\mathbf{p}}\tau} \tag{A.24}$$

Going back to frequency-momentum space, one rewrites the propagator

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \sum_{s} \frac{-s}{2E_{\mathbf{p}}} \frac{1}{i\omega_n - sE_{\mathbf{p}}} \tag{A.25}$$

In terms of fermion propagators, we have

$$S_{\alpha\beta}(\tau, \mathbf{p}) = (\not{p}_s - m)_{\alpha\beta} \tilde{\Delta}(\tau, \mathbf{p}) = \sum_{s=\pm} \frac{s}{2E_{\mathbf{p}}} \tilde{n}(-sE_{\mathbf{p}}) \mathrm{e}^{-sE_{\mathbf{p}}\tau} (\not{p} - m)_{\alpha\beta} \tag{A.26}$$

Comparing (A.19) and (A.24), we see that the frequency sum rules for fermions can be immediately obtained from the rules for bosons, provided we make a substitution $n(E) \to \tilde{n}(E)$. For example, from (A.22), we can read off

$$\begin{aligned} \tilde{S}(i\omega_m, \mathbf{p}) &= \frac{1}{\beta} \sum_n \tilde{\Delta}(i\omega_n, \mathbf{k}) \tilde{\Delta}(i(\omega_n - \omega_m), \mathbf{k} - \mathbf{p}) \\ &= \sum_{s_1,s_2} \frac{s_1 s_2}{4E_1E_2} \frac{1 - \tilde{n}(-s_1E_1) - \tilde{n}(-s_2E_2)}{i\omega_m - s_1E_1 - s_2E_2} \\ &= -\sum_{s_1,s_2} \frac{s_1 s_2}{4E_1E_2} \frac{1 - \tilde{n}(s_1E_1) - \tilde{n}(s_2E_2)}{i\omega_m - s_1E_1 - s_2E_2} \end{aligned} \tag{A.27}$$

More complicated frequency sums can in principle be done with the two methods we introduce here. A collection of results for frequency sums that one shall encounter when evaluating Feynman diagrams can be found in [2].

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